Q1. (30%) Consider the following problem
\[
\min_x 47x_1 + 93x_2 + 17x_3 - 93x_4 \\
s.t. \begin{bmatrix}
-1 & -6 & 1 & 3 \\
-1 & -2 & 7 & 1 \\
0 & 3 & -10 & -1 \\
-6 & -11 & -2 & 12 \\
1 & 6 & -1 & -3 \\
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix}.
\]

(a) Write down the dual problem.
(b) Use CVX to solve the primal and the dual problem to obtain a primal optimal solution \(x^\star\) and a dual optimal solution \(\lambda^\star\), respectively. You are required to send via email your matlab code to the tutor.
(c) Write down the KKT conditions. Then, verify numerically by matlab that \(x^\star\) and \(\lambda^\star\) satisfy the KKT conditions.

Q2. (30%) Implement the subgradient method in matlab to solve the following matrix completion problem.
\[
\min_X \|X\|_* \\
s.t. X_{ij} = A_{ij}, \ (i, j) \in \Omega,
\]
where \(\{A_{ij}\}_{(i, j) \in \Omega}\) is given. Download test data from http://dsp.ee.cuhk.edu.hk/eleg5481/homework/hw5_q2data.mat. The data includes two matrices \(A \in \mathbb{R}^{m \times n}\) and \(I \in \mathbb{R}^{m \times n}\). \(I\) is a zero-one matrix indicating the indices in \(\Omega\). For example, if \(I_{10, 5} = 1\), then \((10, 5)\) belongs to \(\Omega\); if \(I_{20, 3} = 0\), then \((20, 3)\) does not belong to \(\Omega\). Write down the 5 \(\times\) 5 leading principal submatrix of \(X\) obtained by your subgradient method. Also, you are required to send via email your matlab code to the tutor.

Hint 1: A subgradient of the nuclear norm \(\|X\|_*\) is given by \(\sum_{i=1}^r u_i v_i^T\), where \(r\) is the rank of \(X\), and \(u_i\) and \(v_i\) are left and right singular vectors corresponding to positive singular value \(\sigma_r\).

Hint 2: The projection of \(X\) onto the set \(C = \{X \mid X_{ij} = A_{ij}, \ (i, j) \in \Omega\}\) is as follows. Let \(\tilde{X}\) be the projection of \(X\) onto \(C\). We have
\[
\tilde{X}_{ij} = \begin{cases} 
A_{ij}, & (i, j) \in \Omega \\
X_{ij}, & \text{otherwise.}
\end{cases}
\]

Hint 3: You may use the following parameters: stepsize \(\alpha_k = 10/\sqrt{k}\) and number of iterations 5000. You are encouraged to try other parameters as well.

Q3. (20%) Suppose \(a \in \mathbb{R}^n\) with \(a_1 \geq a_2 \geq \ldots \geq a_n > 0\), and \(b \in \mathbb{R}^n\) with \(b_k = 1/a_k\). Derive the KKT conditions for the convex optimization problem
\[
\min_x -\log(a^T x) - \log(b^T x) \\
s.t. x \geq 0, \ 1^T x = 1.
\]
Show that \(x = (1/2, 0, \ldots, 0, 1/2)\) is optimal.
Q4. (a) (10%) Consider the following problem.

\[
\begin{align*}
\min_X & \quad \text{tr}(CX) \\
\text{s.t.} & \quad \text{tr}(AX) \leq 1, \quad \text{tr}(BX) = 1, \\
& \quad X \succeq 0,
\end{align*}
\]  

(1)

where \(A, B, C \in \mathbb{S}^n\) are symmetric matrices. Assume that this problem is strictly feasible. Show that there exists an optimal solution \(X^*\) whose rank satisfies \(\text{rank}(X^*) \leq 1\).

Hint: i) You can use the following matrix decomposition result [Ye-Zhang’03]. Suppose that \(X \in \mathbb{S}^n_+\) has rank \(r\), and \(F \in \mathbb{S}^n\) is given symmetric matrix. Then, there exists a set of vectors \(\{x_1, \ldots, x_r\} \subset \mathbb{R}^n\) such that

\[
X = \sum_{j=1}^{r} x_j x_j^T
\]

and

\[
x_j^T F x_j = \frac{\text{tr}(FX)}{r}, \quad j = 1, \ldots, r.
\]

ii) Use the KKT conditions.

(b) (10%) Consider the S-Lemma, which is widely used in many areas. Let \(F_1, F_2 \in \mathbb{S}^n\), \(g_1, g_2 \in \mathbb{R}^n\), and \(h_1, h_2 \in \mathbb{R}\). Assume that there exists a point \(x_0\) such that \(x_0^T F_1 x_0 + 2g_1^T x_0 + h_1 < 0\). The S-Lemma states that

\[
x^T F_2 x + 2g_2^T x + h_2 \leq 0 \quad \text{for all } x \text{ satisfying } x^T F_1 x + 2g_1^T x + h_1 \leq 0
\]

holds if and only if there exists a \(\lambda \geq 0\) such that

\[
\begin{bmatrix}
F_2 & g_2 \\
g_2^T & h_2
\end{bmatrix} \preceq \lambda \begin{bmatrix}
F_1 & g_1 \\
g_1^T & h_1
\end{bmatrix}.
\]

Prove the S-Lemma by using the result in Q.4(a).

Reference: