ELEG5481 Signal Processing Optimization Techniques
Midterm solution

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Q1. (a) The set C is the convex hull of $\{x_i\}_{i=1}^k$, and thus is convex.
(b) Any function $g(x) = f(x) - (a^T x + b)$ is convex if $f(x)$ is convex. A sublevel set of a convex function is convex.
(c) $\|x\|_q$ is concave on $\mathbf{R}_+^n$ when $0 < q < 1$. Hence C is convex.
(d) For a given w, the set $\{x \mid |\sum_{i=1}^n x_i \exp(\boldsymbol{j}\omega i)|^2 \le 0.1\}$ is the intersection of two half spaces. Hence C is convex.
(e) Use Schur's complement. C can be written as

$$C = \left\{ (X, y) \in \mathbf{S}^n \times \mathbf{R}^n \;\middle|\; \begin{bmatrix} 1 & y^T \\ y & X \end{bmatrix} \succeq 0 \right\}.$$

Note that (X, y) is an affine function of $\begin{bmatrix} 1 & y^T \\ y & X \end{bmatrix}$. Hence C is convex.

Q2. (a) $\|x\|_1$ is convex and y^3 is convex and nondecreasing in $\mathbf{R}_+$. Hence $\|x\|_1^3$ is convex. Therefore $f(x)$ is convex.
(b) $1/x_i$ is concave on $x_i < 0$ for all i. Hence $f(x)$ is concave.
(c) $f(x)$ can be written as $f(x) = -2b^T Ax + b^T b$. Hence $f(x)$ is convex and concave.
(d) $\lambda_{\max}(X)$ is convex, and thus $\lambda_{\max}(-X)$ is convex. $f(X) = -\lambda_{\min}(X) = \lambda_{\max}(-X)$ is convex.
(e) $-\log\det(X)$ is convex in $\mathbf{S}_{++}^n$. $A_0 + A_1 x_1 + \ldots + A_n x_n$ is linear in x. So $f(x)$ is convex.

Q3. (a) Let t be a slack variable, then the problem can be reformulated as

$$\begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & a_1^T x + b_1 + \|Cx - d\|_\infty \le t \\ & a_2^T x + b_2 + \|Cx - d\|_\infty \le t \end{aligned}$$

Let c_i^T denote the ith row of C, then $a_1^T x + b_1 + \|Cx - d\|_\infty \le t$ is the same as $a_1^T x + b_1 + c_i^T x - d \le t$ and $a_1^T x + b_1 - c_i^T x + d \le t$, for all i. Similarly, $a_2^T x + b_2 + \|Cx - d\|_\infty \le t$ is the same as $a_2^T x + b_2 + c_i^T x - d \le t$ and $a_2^T x + b_2 - c_i^T x + d \le t$, for all i. Therefore, we can rewrite the problem as

$$\begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & a_1^T x + b_1 + c_i^T x - d \le t \\ & a_1^T x + b_1 - c_i^T x + d \le t \\ & a_2^T x + b_2 + c_i^T x - d \le t \\ & a_2^T x + b_2 - c_i^T x + d \le t \\ & \text{for } i = 1, \ldots, m. \end{aligned}$$

(b) Let x_0 be a local optimal solution which is not globally optimal. This implies that there exists a point $x_1 \in C$, which is different from x_0, such that $f(x_1) < f(x_0)$. Since x_0 is locally optimal, there exists an $R > 0$ such that $f(x_0) \le f(x)$ for all $x \in C \cap B_R(x_0)$, where $B_R(x_0) = \{x \mid \|x - x_0\|_2 \le R\}$. Construct $x_2 = \theta x_0 + (1-\theta)x_1$, where $0 < \theta < 1$ is a number very closed to one (It suffices to take $\theta \ge 1 - R/\|x_0 - x_1\|_2$). Then x_2 belongs to $C \cap B_R(x_0)$. If we have $f(x_2) < f(x_0)$, then we have contradiction and we are done. To show that $f(x_2) < f(x_0)$, by convexity of f, we have $f(x_2) = f(\theta x_0 + (1-\theta)x_1) \le \theta f(x_0) + (1-\theta) f(x_1) < f(x_0)$.

We will show that there is at most one optimal solution (It may have no solution at all). Let x_1 and x_2 be two different optimal solutions, i.e. $f(x_1) = f(x_2)$. Let θ be an arbitrary point in $(0, 1)$, and $x_3 = \theta x_1 + (1-\theta)x_2$. Then $x_3 \in C$, and $f(x_3) < \theta f(x_1) + (1-\theta)f(x_2) = f(x_1)$ as $f(x)$ is strictly convex. This implies that x_1 is not optimal, which is a contradiction.

Q4. (a) We show this by induction. For $k = 2$, this is obviously true as this is the definition of convex set. Assuming this is true for $k = n$, we need to show that it is true for $k = n + 1$. For $k = n + 1$, y can be written as

$$y = \lambda_{n+1}x_{n+1} + \left(\sum_{i=1}^{n} \lambda_i x_i\right).$$

If $\lambda_{n+1} = 0$, then x is a convex combination of n points, and thus belongs to C. If $\lambda_{n+1} = 1$, then $y = x_{k+1}$ belongs to C. If $0 < \lambda_{n+1} < 1$, we have

$$y = \lambda_{n+1}x_{n+1} + (1 - \lambda_{n+1})\tilde{y},$$

where

$$\tilde{y} = \sum_{i=1}^{n} \frac{\lambda_i}{(1-\lambda_{n+1})} x_i.$$

Note that $\sum_{i=1}^{n} \frac{\lambda_i}{(1-\lambda_{n+1})} = 1$, hence $\tilde{y}$ belongs to C. As y is a convex combination of x_{n+1} and $\tilde{y}$, y belongs to C.

(b) Let us assume that $x_1, \ldots, x_n > 0$. Otherwise, the inequality holds trivially. Consider the function $\log x$ which is convex in $\mathbf{R}_+$. By Jensen's inequality, we have

$$\log\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq \frac{1}{n}\sum_{i=1}^{n} \log x_i,$$

which implies that

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \exp\left(\frac{1}{n}\sum_{i=1}^{n} \log x_i\right) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}.$$

Note that this is the Hadamard's inequality. First assume that all diagonal elements of X are one. Then

$$\det(X) = \prod_{i=1}^{n} \lambda_i \leq \left(\frac{1}{n}\sum_{i=1}^{n} \lambda_i\right)^n = \left(\frac{1}{n}\mathbf{tr}X\right)^n = 1 = \prod_{i=1}^{n} X_{ii}.$$

Now consider a PSD X that has some zeros on the diagonal, i.e. $X_{ii} = 0$ for some i. Then $X_{j,i} = 0$ for $j = 1, \ldots, n$, which implies $\det X = 0$. We also have $\prod_{i=1}^{n} X_{ii} = 0$. Hence the Hadamard's inequality is true.

Consider the case that X is PSD and all diagonal elements of X are positive. Construct a diagonal matrix $D^{-\frac{1}{2}} = \mathbf{Diag}(X_{11}^{-\frac{1}{2}}, \ldots, X_{nn}^{-\frac{1}{2}})$. Then $D^{-\frac{1}{2}}XD^{-\frac{1}{2}}$ is PSD and all diagonal elements are one. By the result above, we have

$$\det(D^{-1}) \times \det(X) = \det(D^{-\frac{1}{2}}XD^{-\frac{1}{2}}) \leq 1.$$

Therefore $\det(X) \leq \det(D) = \prod_{i=1}^{n} X_{ii}$.