ELEG5481 Signal Processing Optimization Techniques Assignment Solution4

Q1. (25%) Formulate the ℓ_4-norm approximation problem

$$\min_{x \in \mathbf{R}^n} \|Ax - b\|_4$$

as a convex QCQP. The matrix $A \in \mathbf{R}^{m \times n}$ and the vector $b \in \mathbf{R}^m$ are given.

Solution: Adding slack variables t_i for $i = 1, \ldots, m$, the problem can be converted to

$$\begin{aligned} \min_{x,t} \quad & \sum_{i=1}^{m} t_i^2 \\ \text{s.t.} \quad & (a_i^T x - b_i)^2 \leq t_i, \quad i = 1, \ldots, m. \end{aligned}$$

which is a convex QCQP.

Q2. (25%) Formulate the following optimization problem as a semidefinte programe.

$$\begin{aligned} \min_{x} \quad & \sup_{\|c\|_2 \leq 1} c^T F(x)^{-1} c \\ \text{s.t.} \quad & F(x) \succ 0, \end{aligned}$$

where

$$F(x) = F_0 + x_1 F_1 + \ldots + x_n F_n$$

with each $F_i \in \mathbf{S}^m$.

Solution: For a given x, $\sup_{\|c\|_2 \leq 1} c^T F(x)^{-1} c$ is the $\lambda_{\max}(F(x)^{-1})$, which is the same as $1/\lambda_{\min}(F(x))$. To minimize $1/\lambda_{\min}(F(x))$, is the same as maximizing $\lambda_{\min}(F(x))$. Therefore, the problem is written as

$$\begin{aligned} \max_{x} \quad & \lambda_{\min}(F(x)) \\ \text{s.t.} \quad & F(x) \succ 0, \end{aligned}$$

which can be again rewritten as

$$\begin{aligned} \max_{x,\lambda} \quad & \lambda \\ \text{s.t.} \quad & F(x) - \lambda I \succeq 0, \\ & F(x) \succ 0. \end{aligned}$$

We can relax this problem a little by allowing $F(x) \succeq 0$.

$$\begin{aligned} \max_{x,\lambda} \quad & \lambda \\ \text{s.t.} \quad & F(x) - \lambda I \succeq 0, \\ & F(x) \succeq 0. \end{aligned}$$

When the optimal solution $\lambda^\star \leq 0$, we know that the original problem is infeasible. And when $\lambda^\star > 0$, an optimal solution $x^\star$ is an optimal solution of the original problem.

Q3. (25%) Consider the following problem

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & \frac{a^T x + b}{c^T x + d} \\ \text{s.t.} \quad & x \succeq 0, \quad x^T 1 \leq 1, \end{aligned}$$

where $c \in \mathbf{R}^n_+$, and $d > 0$. Show that this problem is equivalent to the following problem

$$\begin{aligned} \min_{y \in \mathbf{R}^n, t \in \mathbf{R}} \quad & a^T y + bt \\ \text{s.t.} \quad & c^T y + dt = 1, \\ & y \succeq 0, \quad y^T 1 \leq t, \\ & t \geq 0. \end{aligned}$$

Solution: Given a x which is feasible in the first problem, it can be easily seen that $(y, t) = (x/(c^T x + d), 1/(c^T x + d))$ is an feasible solution of the second problem, and it shares the same objective value of x. This show that the optimal objective value of the first problem is no greater than that of the second problem.

Conversely, given a feasible solution (y, t) of the second problem, we claim that $t > 0$. For otherwise, $t = 0$ and $y = 0$ which violate $c^T y + dt = 1$. It can be easily seen that $x = y/t$ is a feasible solution of the first problem, and shares the same objective value of the second problem. This show that the optimal objective value of the second problem is no greater than that of the first problem.

To conclude, these two problems share the same optimal objective value.

Q4. (a) (7%) For $u \in \mathbf{R}^K_{++}$, the arithmetic mean $H_1(u)$, geometric mean $H_2(u)$, harmonic mean $H_3(u)$, and the minimum value $H_4(u)$ are defined as

$$H_1(u) = \frac{1}{K} \sum_{k=1}^{K} u_k, \qquad H_2(u) = \left(\prod_{k=1}^{K} u_k \right)^{1/K},$$

$$H_3(u) = K \left(\sum_{k=1}^{K} u_k^{-1} \right)^{-1}, \qquad H_4(u) = \min_{k=1,\ldots,K} u_k.$$

Show that $H_1(u) \geq H_2(u) \geq H_3(u) \geq H_4(u)$. Hint: You don't need to show $H_1(u) \geq H_2(u)$ which you have seen in the midterm exam. For $H_2(u) \geq H_3(u)$, try using $H_1(u) \geq H_2(u)$.

(b) (9%) Convert the following optimization problem to a quasi-convex problem.

$$\begin{aligned} \max_{s, u \in \mathbf{R}^K} \quad & H_4(u) \\ \text{s.t.} \quad & u_k = \ln\left(1 + \frac{s_k}{\sigma_k + \sum_{j \neq k} \alpha_{k,j} s_j}\right), \qquad && k = 1, \ldots, K, \\ & 0 \leq s_k \leq P_k, && k = 1, \ldots, K, \end{aligned}$$

where all $\sigma_k, \alpha_{k,j}$ and P_k are given positive numbers. Hint: consider generalized linear-fractional programming.

(c) (9%) Convert the following optimization problem to a convex problem.

$$\begin{aligned} \max_{s, u \in \mathbf{R}^K} \quad & H_2(u) \\ \text{s.t.} \quad & u_k = \ln\left(1 + \frac{s_k}{\sigma_k + \sum_{j \neq k} \alpha_{k,j} s_j}\right), \qquad && k = 1, \ldots, K, \\ & 0 \leq s_k \leq P_k, && k = 1, \ldots, K, \end{aligned}$$

where all $\sigma_k, \alpha_{k,j}$ and P_k are given positive numbers. Hint: The following functions are convex:

$$f(x) = \ln \sum_{k=1}^{K} a_k e^{x_k}, \quad \mathbf{dom}\ f = \mathbf{R}_{++}^K, \qquad \text{when } a_k \geq 0 \text{ for } k = 1, \ldots, K,$$
$$g(t) = \ln(e^{1/t} - 1), \quad \mathbf{dom}\ g = \mathbf{R}_{++}.$$

Solution:

(a) By $H_1(u) \geq H_2(u)$, we have

$$H_2(u)/H_3(u) = \frac{1}{K}\left(\prod_{k=1}^{K} u_k\right)^{1/K}\left(\sum_{k=1}^{K} u_k^{-1}\right) = \frac{1}{K}\sum_{k=1}^{K}\frac{\left(\prod_{j=1}^{K} u_j\right)^{1/K}}{u_k} \geq \frac{\prod_{j=1}^{K} u_j}{\prod_{k=1}^{K} u_k} = 1.$$

It is easy to show $H_3(u) \geq H_4(u)$. Indeed, we have

$$H_3(u) \geq K\left(K u_{\min}^{-1}\right)^{-1} = u_{\min} = H_4(u),$$

where $u_{\min} = \min_{k=1,\ldots,K} u_k$.

(b) It is equivalent to maximize the minimum u_k. Therefore we can recast the problem as

$$\begin{aligned} \max_{s \in \mathbf{R}^K} \quad & \min_{k=1,\ldots,K} \frac{s_k}{\sigma_k + \sum_{j \neq k} \alpha_{k,j} s_j} \\ \text{s.t.} \quad & 0 \leq s_k \leq P_k, \qquad\qquad k = 1, \ldots, K. \end{aligned}$$

The constraints are convex obviously. We will show that the objective function is quasiconcave. The superlevel set is

$$\begin{aligned} S_\alpha &= \{s \mid \frac{s_k}{\sigma_k + \sum_{j \neq k} \alpha_{k,j} s_j} \geq \alpha,\ k = 1, \ldots, K.\} \\ &= \left\{ s \mid s_k - \alpha\left(\sigma_k + \sum_{j \neq k} \alpha_{k,j} s_j\right) \geq 0,\ k = 1, \ldots, K. \right\} \end{aligned}$$

This is the intersection of half spaces, which is convex.

(c) There are some typos in the hint. There is no mark reduction for this reason. The correct hint is

$$f(x) = \ln \sum_{k=1}^{K} a_k e^{x_k}, \quad \mathbf{dom}\ f = \mathbf{R}^K, \qquad \text{when } a_k \geq 0 \text{ for } k = 1, \ldots, K,$$
$$g(t) = \ln(e^{e^t} - 1), \quad \mathbf{dom}\ g = \mathbf{R}.$$

Taking log of the objective function, discarding the constant $1/K$, and changing the equality constraint to inequality constraint (why we can do it?), the problem can be rewritten as

$$\begin{aligned} \max_{s,u \in \mathbf{R}^K} \quad & \sum_{k=1}^{K} \ln u_k \\ \text{s.t.} \quad & u_k \leq \ln\left(1 + \frac{s_k}{\sigma_k + \sum_{j \neq k} \alpha_{k,j} s_j}\right), \qquad k = 1, \ldots, K, \\ & 0 \leq s_k \leq P_k, \qquad\qquad k = 1, \ldots, K, \end{aligned}$$

Using variable transformation $u_k = \exp t_k$, the problem is rewritten as

$$\begin{aligned} \max_{s,t \in \mathbf{R}^K} \quad & \sum_{k=1}^{K} \ln e^{t_k} \\ \text{s.t.} \quad & e^{t_k} \leq \ln\left(1 + \frac{s_k}{\sigma_k + \sum_{j \neq k} \alpha_{k,j} s_j}\right) \\ & 0 \leq s_k \leq P_k, \quad k = 1, \ldots, K. \end{aligned}$$

The first constraint is the same as

$$\frac{\sigma_k + \sum_{j \neq k} \alpha_{kj} s_j}{s_k} \leq \frac{1}{\exp(\exp(t_k)) - 1}$$

Using the variable transformation $s_j = \exp(y_j)$ and taking log, the first constraint can be rewrite as

$$\ln \left(\sigma_k \exp(-y_k) + \sum_{j \neq k} \alpha_{kj} \exp(y_j - y_k) \right) + \ln(\exp(\exp(t_k)) - 1) \leq 0.$$

which is convex.
Therefore, we can write the problem as

$$\begin{aligned} \max_{y, t \in \mathbf{R}^K} \quad & \sum_{k=1}^{K} t_k \\ \text{s.t.} \quad & \ln \left(\sigma_k \exp(-y_k) + \sum_{j \neq k} \alpha_{kj} \exp(y_j - y_k) \right) + \ln(\exp(\exp(t_k)) - 1) \leq 0. \\ & y_k \leq \ln P_k, \\ & k = 1, \ldots, K. \end{aligned}$$

It can be check that $\ln(\exp(\exp(t)) - 1)$ is convex. Therefore, this is a convex problem.