ELEG5481 Signal Processing Optimization Techniques
Assignment Solution 2

Q1. (25%) Are the following sets convex? Give a brief justification for each of the following cases:
(a) $C = \{x \mid a_1^T x \geq b_1 \text{ or } a_2^T x \geq b_2\}$
(b) $C = \{x \in \mathbf{R}^n \mid \|x\|_2 = 1, \sum_{i=1}^n x_i = \sqrt{n}\}$
(c) $C = \{x | (a^T x + b)/(c^T x + d) \geq 1, \quad c^T x + d > 0\}$.
(d) $C = \{x | \max_{i=1,\ldots,n} x_i \leq a\}$, where $a \in \mathbf{R}$.

Solution:
(a) No in general. Union of two convex sets may not be convex. Let us write $C_1 = \{x \mid a_1^T x \geq b_1\}$ and $C_2 = \{x \mid a_2^T x \geq b_2\}$. In the cases that $C_1 \subset C_2$, or $C_2 \subset C_1$, or $C_1 \cup C_2 = \mathbf{R}^n$, C is convex. Otherwise, C is not covex.
(b) C is convex. It could be seen that $C = \{\frac{1}{\sqrt{n}}1\}$ which is a singleton.
(c) C is convex. C can be written as $\{x \mid a^T x + b \geq c^T x + d, \quad c^T x + d > 0\}$ which is the intersection of two half spaces.
(d) C is convex. C can be written as $\{x \mid e_i^T x \leq a, \; i = 1, \ldots, n\}$ which is the intersection of half spaces.

Q2. (25%) Show that the following sets are convex:
(a) $C = \{x \in \mathbf{R}^n \mid \sum_{i=1}^n |x_i| \leq 1\}$.
(b) $C = \{x \mid \|x - c\| \leq a^T x + b\}$, where $\|\cdot\|$ is a norm.
(c) $C = \{x \mid B(x, a) \subset S\}$, where $a \geq 0$, S is convex set, and $B(x, a) = \{y \mid \|y - x\| \leq a\}$.
(d)

$$C = \{r \in \mathbf{R}^n \mid T(r) \succeq 0\}$$

where $T : \mathbf{R}^n \to \mathbf{S}^n$ is given by

$$T(r) = \begin{bmatrix} r_1 & r_2 & \cdots & r_{n-1} & r_n \\ r_2 & r_1 & r_2 & & r_{n-1} \\ \vdots & r_2 & \ddots & \ddots & \vdots \\ r_{n-1} & & \ddots & \ddots & r_2 \\ r_n & r_{n-1} & & r_2 & r_1 \end{bmatrix}.$$

Solution:
(a) C can be written as $C = \bigcap_{y:|y_i|=1,\forall i}\{x \mid x^T y \leq 1\}$ which is the intersection of half spaces.
(b) Let x,$y \in C$ and $0 \leq \theta \leq 1$. Then we have

$$\begin{aligned} \|\theta x + (1-\theta)y - c\| &= \|\theta(x - c) + (1-\theta)(y - c)\| \leq \theta\|x - c\| + (1-\theta)\|y - c\| \\ &\leq \theta(a^T x + b) + (1-\theta)(a^T y - b) = a^T(\theta x + (1-\theta)y) - b \end{aligned}$$

(c) Let x_1, $x_2 \in C$ and $0 \leq \theta \leq 1$. We need to show that $B(\theta x_1 + (1-\theta)x_2, a)$ is a subset of S, or equivalently if y satisfies

$$\|y - (\theta x_1 + (1-\theta)x_2)\| \leq a,$$

then y belongs to S. We decompose y in the form of $y = \theta y_1 + (1-\theta)y_2$, where $y_1 = y - (1-\theta)(x_2 - x_1)$ and $y_2 = y + \theta(x_2 - x_1)$. If y_1 and y_2 belongs to S, then y belongs to S, as S is convex. Indeed, we have

$$\|y_1 - x_1\| = \|y - x_1 - (1-\theta)(x_2 - x_1)\| = \|y - (\theta x_1 + (1-\theta)x_2)\| \le a,$$
$$\|y_2 - x_2\| = \|y - x_2 + \theta(x_2 - x_1)\| = \|y - (\theta x_1 + (1-\theta)x_2)\| \le a.$$

Therefore $y_1 \in B(x_1, a) \subset S$ and $y_2 \in B(x_2, a) \subset S$.

(d) $T(r)$ is an affine mapping of r. C is the inverse image of the PSD cone under the mapping $T(r)$. So C is convex.

Q3. (25%) Let $x_0, \ldots, x_k \in \mathbf{R}^n$. Consider the set of points that are closer (in Euclidean norm) to x_0 than the other x_i, i.e.,

$$V = \{x \in \mathbf{R}^n \mid \|x - x_0\|_2 \le \|x - x_i\|_2,\ i = 1, \ldots, K\}.$$

V is called the Voronoi region around x_0 with respect to $x_1, \ldots, x_K$.

(a) Show that V is a polyhedron, and thus convex. Express V in the form $V = \{x \mid Ax \preceq b\}$.

(b) Conversely, given a polyhedron P with nonempty interior, show how to find $x_0, \ldots, x_K$ so that the polyhedron is the Voronoi region of x_0 with respect to $x_1, \ldots, x_K$.

Solution:

(a) We have

$$\|x - x_0\|_2 \le \|x - x_i\|_2 \Longleftrightarrow 2x^T(x_0 - x_i) \le x_0^T x_0 - x_i^T x_i.$$

Let a_i^T denote the ith row of A. Then it suffices to take $a_i = 2(x_0 - x_i)$ and $b_i = x_0^T x_0 - x_i^T x_i$. Hence, V is a polyhedron and thus convex.

(b) Conversely, given a polyhedron $P = \{x \mid Ax \le b\}$ with nonempty interior, take x_0 to be a point in the interior of P, i.e. $Ax < b$. Then take $x_i = x_0 + 2(\frac{b - a^T x_0}{a^T a})a$. It can be easily shown that

$$a^T x \le b \Longleftrightarrow \|x - x_0\|_2 \le \|x - x_i\|_2.$$

Therefore $P = \{x \in \mathbf{R}^n \mid \|x - x_0\|_2 \le \|x - x_i\|_2,\ i = 1, \ldots, K\}$.

Q4. (25%) Let K be a cone. The set

$$K^* = \{y \mid x^T y \ge 0, \text{ for all } x \in K\}$$

is called the dual cone of K. Show that K^* is always a convex cone. Moreover, determine the dual cones of the following sets:

(a) $K = \mathbf{R}^n_+$.

(b) $K = \mathbf{S}^n_+$ (in which case the dual cone should be written as $K^* = \{Y \mid \mathbf{tr}(XY) \ge 0, \text{ for all } X \in K\}$).

Solution:

(a) The dual cone is $\mathbf{R}^n_+$. Obviously, if $y \in \mathbf{R}^n_+$, then $x^T y \ge 0$ for all $x \in \mathbf{R}^n_+$. Therefore $\mathbf{R}^n_+ \subset (\mathbf{R}^n_+)^*$. Conversely, we need to show that if $y^T x \ge 0$ for all $x \in \mathbf{R}^n_+$, then $y \in \mathbf{R}^n_+$. This result can be obtained by setting $x = e_i$, for $i = 1, \ldots, n$, respectively.

(b) The dual cone is $\mathbf{S}^n_+$. For a proof, see Q6 in assignment 1.