ELEG5481 Signal Processing Optimization Techniques
Assignment Solution 1

Feb. 21, 2013

Q1. Show that following functions are norms.
(a) $f(x) = \|Tx\|$, where$\|\cdot\|$ is a norm, $T \in \mathbf{C}^{n\times n}$ is nonsingular, and $x \in \mathbf{C}^n$. In particular, $f(x) = \sqrt{x^T P x}$ with $P \succ 0$ is a norm.
(b) $f(X) = \max_{1\le i\le m}\{\sum_{j=1}^n |X_{ij}|\}$, where $X \in \mathbf{C}^{m\times n}$.
(c) $f(X) = \sup\{\|Xu\|_a \mid \|u\|_b \le 1\}$, where $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms, and $X \in \mathbf{C}^{m\times n}$.

Solution:
(a) 1.) $\|Tx\| \ge 0$.
2.) $\|Tx\| = 0$ i.f.f. $Tx = 0$, since $\|\cdot\|$ is a norm. Moreover, $Tx = 0$ i.f.f. $x = 0$ since T is nonsingular.
3.) $\|T(cx)\| = |c|\|Tx\|$.
4.) $\|T(x+y)\| = \|Tx + Ty\| \le \|Tx\| + \|Ty\|$.

(b) 1.)Obviously $f(X) \ge 0$.
2.) Still obviously $\max_{1\le i\le m}\{\sum_{j=1}^n |X_{ij}|\} = 0$ i.f.f. $X_{ij} = 0$ for all i, j.
3.) $f(cX) = \max_{1\le i\le m}\{\sum_{j=1}^n |cX_{ij}|\} = |c|\max_{1\le i\le m}\{\sum_{j=1}^n |X_{ij}|\} = |c|f(X)$.
4.)

$$\begin{aligned} f(X+Y) &= \max_{1\le i\le m}\{\sum_{j=1}^n |X_{ij}+Y_{ij}|\} \le \max_{1\le i\le m}\{\sum_{j=1}^n (|X_{ij}| + |Y_{ij}|)\} = \max_{1\le i\le m}\{\sum_{j=1}^n |X_{ij}| + \sum_{j=1}^n |Y_{ij}|)\} \\ &\le \max_{1\le i\le m}\{\sum_{j=1}^n |X_{ij}|\} + \max_{1\le i\le m}\{\sum_{j=1}^n |Y_{ij}|)\} = f(X) + f(Y). \end{aligned}$$

(c) 1.) $f(X) \ge \|X0\|_a = 0$, where 0 is the zero vector.
2.) Let $\mathbf{0}$ denote the zero matrix. $f(\mathbf{0}) = \sup\{\|0\|_a \mid \|u\|_b \le 1\} = 0$. Conversely, $f(X) = 0$ implies that $\|Xu\|_a = 0$ for all u such that $\|u\|_b \le 1$. By setting $u = e_i/\|e_i\|_b$, where e_i is the all zero vector except the ith entry being one, we have that the ith column of X is a zero vector.
3.) $f(cX) = \sup\{\|cXu\|_a \mid \|u\|_b \le 1\} = |c|\sup\{\|Xu\|_a \mid \|u\|_b \le 1\} = |c|f(X)$.
4.) $f(X+Y) = \sup\{\|(X+Y)u\|_a \mid \|u\|_b \le 1\} \le \sup\{\|(Xu\|_a \mid \|u\|_b \le 1\} + \sup\{\|Yu\|_a \mid \|u\|_b \le 1\} = f(X) + f(Y)$.

Q2. Prove that

$$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2.$$

Solution:

$$\|x\|_1 = \sum_{i=1}^n |x_i| \times 1 \le \sqrt{\sum_{i=1}^n |x_i|^2}\sqrt{\sum_{i=1}^n 1} = \sqrt{n}\|x\|_2,$$

$$\|x\|_1^2 = (\sum_{i=1}^{n} |x_i|)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |x_i||x_j| = \sum_{i=1}^{n} |x_i|^2 + \sum_{i\neq j} |x_i||x_j| \geq \sum_{i=1}^{n} |x_i|^2 = \|x\|_2^2$$
$$\Longrightarrow \|x\|_1 \geq \|x\|_2.$$

Q3. The matrix p-norm is defined as

$$\|X\|_p = \max\{\|Xu\|_p \mid \|u\|_p \leq 1\}.$$

Show the following results
(a) $\|X\|_\infty = \max_{1\leq i\leq m}\{\sum_{j=1}^n |X_{ij}|\}$.
(b) $\|X\|_2 = \sigma_{\max}(X) = \sqrt{\lambda_{\max}(X^TX)}$, where $\sigma_{\max}(X)$ denotes the largest singular value of X, and $\lambda_{\max}(A)$ the largest eigenvalue of A.

Solution:
(a) Let x_i^T denote the ith row of X. Then we have

$$\begin{aligned}
\|X\|_\infty &= \max_{\|u\|_\infty \leq 1} \{\|Xu\|_\infty\} \\
&= \max_{\|u\|_\infty \leq 1} \left\{ \max_{i=1,\dots,m} |x_i^T u| \right\} \\
&= \max_{i=1,\dots,m} \left\{ \max_{\|u\|_\infty \leq 1} |x_i^T u| \right\} \\
&= \max_{i=1,\dots,m} \left\{ \sum_{j=1}^{n} |X_{ij}| \right\}
\end{aligned}$$

(b)
$$\|X\|_2^2 = \sup\{\|Xu\|_2^2 \mid \|u\|_2 \leq 1\} = \sup\{u^TX^TXu \mid \|u\|_2 \leq 1\}$$

Since
$$\frac{u^TX^TXu}{u^Tu} \leq \lambda_{max}(X^TX)$$
and the upper bound is achieved when u is the principal eigenvector. Therefore,

$$\|X\|_2^2 = \lambda_{max}(X^TX).$$

Let $X^TX = Q\Lambda Q^T$, and $X = U\Sigma V^T$. Since

$$X^TX = V\Sigma^TU^TU\Sigma V^T = V\Sigma^2V^T,$$

we have
$$\sigma_i^2 = \lambda_i(X^TX) \Leftrightarrow \sigma_i = \sqrt{\lambda_i(X^TX)}.$$

Q4. Verify the following: Given that $A \in \mathbf{C}^{n\times n}$ is Hermitian,
(a) $\|A\|_F^2 = \sum_{i=1}^n \lambda_i^2$.
(b) $\det(A) = \prod_{i=1}^n \lambda_i$
(c) $\mathrm{tr}(A) = \sum_{i=1}^n \lambda_i$.

Solution: Use eigendecomposition $A = Q\Lambda Q^H$.

(a)

$$\|A\|_F^2 = \text{tr}(A^H A) = \text{tr}(Q\Lambda Q^H Q\Lambda Q^H) = \text{tr}(\Lambda^2) = \sum_i \lambda_i^2$$

(b)

$$\det(A) = \det(Q\Lambda Q^H) = \det(Q^H Q\Lambda) = \det(\Lambda) = \prod_i \lambda_i$$

(c)

$$\text{tr}(A) = \text{tr}(Q\Lambda Q^H) = \text{tr}(Q^H Q\Lambda) = \text{tr}(\Lambda) = \sum_i \lambda_i$$

Q5. Show the following results

(a) Suppose $A \in \mathbf{R}^{n\times n}$ is positive semidefinite. If $A_{i,i} = 0$ for some i, then $A_{j,i}$ and $A_{i,j}$ are zeros for all $j = 1, \dots, n$.

(b) Suppose $A \in \mathbf{C}^{n\times n}$ is positive semidefinite. If $A_{i,i} = 1$ for $i = 1, \dots, n$, then $|A_{i,j}| \leq 1$ for any i, j.

Solution:

(a) Consider an x whose ith entry is x_i, jth entry one, and all other entries zeros. Then we have $x^T Ax = A_{j,j} + 2A_{i,j}x_i \geq 0$ for any x_i. But this is impossible unless $A_{i,j} = 0$. To see why, suppose $A_{i,j} > 0$. Then for $x_i < -A_{j,j}/2A_{i,j}$, we have $x^T Ax < 0$. Similarly, $A_{i,j}$ can not be negative.

(b) Consider an x whose ith entry is 1, jth entry is $e^{\mathbf{j}\theta}$, and other entries zeros, where $\mathbf{j} = \sqrt{-1}$ and $\theta \in \mathbf{R}$. Then we have $x^H Ax = A_{i,i} + A_{j,j} + 2\mathcal{R}\{A_{i,j}e^{\mathbf{j}\theta}\} = 2 + 2\mathcal{R}\{A_{i,j}e^{\mathbf{j}\theta}\} \geq 0$. Thus $|A_{i,j}| \leq 1$.

Q6. (10%) Let $X \in \mathbf{S}^n$. Show that

$$\text{tr}(XY) \geq 0 \quad \forall\, Y \in \mathbf{S}_+^n$$

if and only if X is PSD.

Solution: The "if" part. Let $X = Q\Lambda Q^T$ denote the EVD. Then we have

$$\text{tr}XY = \text{tr}Q\Lambda Q^T Y = \text{tr}\Lambda(Q^T YQ) = \sum_{i=1}^n \lambda_i [Q^T YQ]_i$$

Since Y is PSD, so is $Q^T YQ$, and thus $[Q^T YQ]_i \geq 0$. Therefore, $\text{tr}XY \geq 0$.

The "only if" part. We have

$$\text{tr}XY = \text{tr}Q\Lambda Q^T Y = \text{tr}\Lambda(Q^T YQ) \geq 0, \text{ for all } Y \in \mathbf{S}_+^n.$$

Setting $Y = Qe_i e_i^T Q^T$, we have $\text{tr}XY = \lambda_i \geq 0$.