ELEG5481 Signal Processing Optimization Techniques
Assignment 5

Apr. 22, 2013

Deadline: 5:00pm, May. 6, 2013

Q1. (30%) Consider the following problem

$$\begin{aligned} \min_x \;& 47x_1 + 93x_2 + 17x_3 - 93x_4 \\ \text{s.t.} \;& \begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \preceq \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix}. \end{aligned}$$

(a) Write down the dual problem.

(b) Use CVX to solve the primal and the dual problem to obtain a primal optimal solution $x^\star$ and a dual optimal solution $\lambda^\star$, respectively. You are required to send via email your matlab code to the tutor.

(c) Write down the KKT conditions. Then, verify numerically by matlab that $x^\star$ and $\lambda^\star$ satisfy the KKT conditions.

Q2. (30%) Implement the subgradient method in matlab to solve the following matrix completion problem.

$$\begin{aligned} \min_X \;& \|X\|_* \\ \text{s.t.} \;& X_{ij} = A_{ij}, \quad (i,j) \in \Omega, \end{aligned}$$

where $\{A_{ij}\}_{(i,j)\in\Omega}$ is given. Download test data from `http://dsp.ee.cuhk.edu.hk/eleg5481/homework/hw5_q2data.mat`. The data includes two matrices $A \in \mathbf{R}^{m\times n}$ and $I \in \mathbf{R}^{m\times n}$. I is a zero-one matrix indicating the indices in Ω. For example, if $I_{10,5} = 1$, then $(10,5)$ belongs to Ω; if $I_{20,3} = 0$, then $(20,3)$ does not belong to Ω. Write down the 5×5 leading principal submatrix of X obtained by your subgradient method. Also, you are required to send via email your matlab code to the tutor.

Hint 1: A subgradient of the nuclear norm $\|X\|_*$ is given by $\sum_{i=1}^r u_i v_i^T$, where r is the rank of X, and u_i and v_i are left and right singular vectors corresponding to positive singular value σ_r.

Hint 2: The projection of X onto the set $C = \{X \mid X_{ij} = A_{ij},\ \ (i,j) \in \Omega\}$ is as follows. Let $\tilde{X}$ be the projection of X onto C. We have

$$\tilde{X}_{ij} = \begin{cases} A_{ij}, & (i,j) \in \Omega \\ X_{ij}, & \text{otherwise.} \end{cases}$$

Hint 3: You may use the following parameters: stepsize $\alpha_k = 10/\sqrt{k}$ and number of iterations 5000. You are encouraged to try other parameters as well.

Q3. (20%) Suppose $a \in \mathbf{R}^n$ with $a_1 \geq a_2 \geq \ldots \geq a_n > 0$, and $b \in \mathbf{R}^n$ with $b_k = 1/a_k$. Derive the KKT conditions for the convex optimization problem

$$\begin{aligned} \min_x \;& -\log(a^T x) - \log(b^T x) \\ \text{s.t.} \;& x \succeq 0, \quad \mathbf{1}^T x = 1. \end{aligned}$$

Show that $x = (1/2, 0, \ldots, 0, 1/2)$ is optimal.

Q4. (a) (10%) Consider the following problem.

$$\begin{aligned} \min_X \ & \mathrm{tr}(CX) \\ \text{s.t. } & \mathrm{tr}(AX) \le 1, \ \mathrm{tr}(BX) = 1, \\ & X \succeq 0, \end{aligned} \tag{1}$$

where $A, B, C \in \mathbf{S}^n$ are symmetric matrices. Assume that this problem is strictly feasible. Show that there exists an optimal solution $X^\star$ whose rank satisfies $\mathrm{rank}(X^\star) \le 1$.

Hint: i) You can use the following matrix decomposition result [Ye-Zhang'03]. Suppose that $X \in \mathbf{S}^n_+$ has rank r, and $F \in \mathbf{S}^n$ is given symmetric matrix. Then, there exists a set of vectors $\{x_1, \ldots, x_r\} \subset \mathbf{R}^n$ such that

$$X = \sum_{j=1}^{r} x_j x_j^T$$

and

$$x_j^T F x_j = \frac{\mathbf{tr}(FX)}{r}, \quad j = 1, \ldots, r.$$

ii) Use the KKT condtions.

(b) (10%) Consider the S-Lemma, which is widely used in many areas. Let $F_1, F_2 \in \mathbf{S}^n$, $g_1, g_2 \in \mathbf{R}^n$, and $h_1, h_2 \in \mathbf{R}$. Assume that there exists a point x_0 such that $x_0 F_1 x_0 + 2g_1^T x_0 + h_1 < 0$. The S-Lemma states that

$$x^T F_2 x + 2g_2^T x + h_2 \le 0 \quad \text{for all } x \text{ satisfying } x^T F_1 x + 2g_1^T x + h_1 \le 0$$

holds if and only if there exists a $\lambda \ge 0$ such that

$$\begin{bmatrix} F_2 & g_2 \\ g_2^T & h_2 \end{bmatrix} \preceq \lambda \begin{bmatrix} F_1 & g_1 \\ g_1^T & h_1 \end{bmatrix}.$$

Prove the S-Lemma by using the result in Q.4(a).

Reference:

Y. Ye and S. Zhang, "New results on quadratic minimization," SIAM J. Optim., 2003.