ELEG5481 Signal Processing Optimization Techniques
Assignment 4

Mar. 26, 2013

Deadline: 5:00pm, Apr. 8, 2013

Q1. (25%) Formulate the ℓ_4-norm approximation problem

$$\min_{x \in \mathbf{R}^n} \|Ax - b\|_4$$

as a convex QCQP. The matrix $A \in \mathbf{R}^{m \times n}$ and the vector $b \in \mathbf{R}^m$ are given.

Q2. (25%) Formulate the following optimization problem as a semidefinte programe.

$$\begin{aligned} \min_{x} \quad & \sup_{\|c\|_2 \leq 1} c^T F(x)^{-1} c \\ \text{s.t.} \quad & F(x) \succ 0, \end{aligned}$$

where

$$F(x) = F_0 + x_1 F_1 + \ldots + x_n F_n$$

with each $F_i \in \mathbf{S}^m$.

Q3. (25%) Consider the following problem

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & \frac{a^T x + b}{c^T x + d} \\ \text{s.t.} \quad & x \succeq 0, \quad x^T 1 \leq 1, \end{aligned}$$

where $c \in \mathbf{R}^n_+$, and $d > 0$. Show that this problem is equivalent to the following problem

$$\begin{aligned} \min_{y \in \mathbf{R}^n, t \in \mathbf{R}} \quad & a^T y + bt \\ \text{s.t.} \quad & c^T y + dt = 1, \\ & y \succeq 0, \quad y^T 1 \leq t, \\ & t \geq 0. \end{aligned}$$

Q4. (a) (7%) For $u \in \mathbf{R}^K_{++}$, the arithmetic mean $H_1(u)$, geometric mean $H_2(u)$, harmonic mean $H_3(u)$, and the minimum value $H_4(u)$ are defined as

$$H_1(u) = \frac{1}{K} \sum_{k=1}^{K} u_k, \qquad H_2(u) = \left(\prod_{k=1}^{K} u_k \right)^{1/K},$$

$$H_3(u) = K \left(\sum_{k=1}^{K} u_k^{-1} \right)^{-1}, \qquad H_4(u) = \min_{k=1,\ldots,K} u_k.$$

Show that $H_1(u) \geq H_2(u) \geq H_3(u) \geq H_4(u)$. Hint: You don't need to show $H_1(u) \geq H_2(u)$ which you have seen in the midterm exam. For $H_2(u) \geq H_3(u)$, try using $H_1(u) \geq H_2(u)$.

(b) (9%) Convert the following optimization problem to a quasi-convex problem.

$$\begin{aligned} \max_{s, u \in \mathbf{R}^K} \quad & H_4(u) \\ \text{s.t.} \quad & u_k = \ln\left(1 + \frac{s_k}{\sigma_k + \sum_{j \neq k} \alpha_{k,j} s_j}\right), \qquad k = 1, \ldots, K, \\ & 0 \leq s_k \leq P_k, \qquad k = 1, \ldots, K, \end{aligned}$$

where all $\sigma_k, \alpha_{k,j}$ and P_k are given positive numbers. Hint: consider generalized linear-fractional programming.

(c) (9%) Convert the following optimization problem to a convex problem.

$$\begin{aligned}
\max_{s,u \in \mathbf{R}^K} \quad & H_2(u) \\
\text{s.t.} \quad & u_k = \ln\left(1 + \frac{s_k}{\sigma_k + \sum_{j \neq k} \alpha_{k,j} s_j}\right), \qquad k = 1, \ldots, K, \\
& 0 \leq s_k \leq P_k, \qquad k = 1, \ldots, K,
\end{aligned}$$

where all $\sigma_k, \alpha_{k,j}$ and P_k are given positive numbers. Hint: The following functions are convex:

$$\begin{aligned}
f(x) &= \ln \sum_{k=1}^{K} a_k e^{x_k}, \quad \textbf{dom}\ f = \mathbf{R}^K_{++}, \qquad \text{when } a_k \geq 0 \text{ for } k = 1, \ldots, K, \\
g(t) &= \ln(e^{1/t} - 1), \quad \textbf{dom}\ g = \mathbf{R}_{++}.
\end{aligned}$$