ELEG5481 Signal Processing Optimization Techniques
Assignment 3

Feb. 28, 2013

Deadline: 5:00pm, Mar. 11, 2013

Q1. (40%) The following functions are either convex, concave, or neither convex nor concave. Identify their convexity/concavity, and provide your answer with a proof.

(a)

$$f(x) = \max\{\|APx - b\| \ \mid \ P \text{ is a permutation matrix } \}$$

with $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$. Note that a permutation matrix P is a square matrix that has exactly one entry 1 in each row and column and 0s elsewhere.

(b)

$$f(x) = \|Ax - b\|_2^2 - \gamma\|x\|_2^2$$

where $\mathbf{dom} f = \mathbf{R}^n$, $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, and $\gamma > 0$.

(c)

$$f(X) = \lambda_{\min}(X) = \inf_{y\in\mathbf{R}^n, y\neq 0} \frac{y^T X y}{y^T y}, \quad \mathbf{dom} f = \mathbf{S}^n$$

(d)

$$f(x) = \int_0^{2\pi} \log p(x,\omega) d\omega,$$

where

$$p(x,\omega) = x_1 + x_2\cos(\omega) + x_3\cos(2\omega) + \ldots x_n\cos((n-1)\omega),$$

and $\mathbf{dom} f = \{x \mid p(x,\omega) > 0, 0 \le \omega \le 2\pi\}$. (Note that $\log(\cdot)$ is the natural log function.)

(e) The difference between the maximum and minimum value of a polynomial on a given interval, as a function of its coefficients:

$$f(x) = \sup_{t\in[a,b]} p(t) - \inf_{t\in[a,b]} p(t),$$

where $p(t) = x_1 + x_2 t + x_3 t^2 + \ldots + x_n t^{n-1}$, and a and b are real constants with $a < b$.

(f)

$$f(x) = \sum_{i=1}^m e^{-1/f_i(x)}, \quad \mathbf{dom} f = \{x \ \mid f_i(x) < 0, \ i = 1, \ldots, m\}.$$

where the functions f_i are convex.

Q2. (15%) Let $p(x)$ and $q(x)$ be functions where $p(x) > 0$ for $x \in S$, $q(x) > 0$ for $x \in S$, $\int_S p(x)dx = 1$ and $\int_S q(x)dx = 1$. Show that

$$\int_S p(x)\log q(x) dx \le \int_S p(x)\log p(x) dx.$$

Q3. (15%) Show that the following function is convex.

$$f(x) = x^T(A(x))^{-1}x, \qquad \mathbf{dom} f = \{x \ \mid \ A(x) \succ 0\},$$

where $A(x) = A_0 + A_1 x_1 + \ldots + A_n x_n \in \mathbf{S}^n$, and $A_i \in \mathbf{S}^n$, $i = 1, \ldots, n$. Hint: You are allowed to use a special form of Schur complement, described as follows: Suppose $A \succ 0$. Then

$$\begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \succeq 0 \Longleftrightarrow c - b^T A^{-1} b \ge 0.$$

Q4. (15%) Let $f_0, \ldots, f_n : \mathbf{R} \to \mathbf{R}$ be given continuous functions. We consider the problem of approximating f_0 as linear combination of $f_1, \ldots, f_n$. For $x \in \mathbf{R}^n$, we say that $f = x_1 f_1 + \ldots + x_n f_n$ approximates f_0 with tolerance $\epsilon > 0$ over the interval $[0, T]$ if $|f(t) - f_0(t)| \leq \epsilon$ for $0 \leq t \leq T$. Now we choose a fixed tolerance $\epsilon > 0$ and define the approximation width as the largest T such that f approximates f_0 over the interval $[0, T]$:

$$W(x) = \sup\{T \mid |x_1 f_1(t) + \ldots + x_n f_n(t) - f_0(t)| \leq \epsilon \text{ for all } 0 \leq t \leq T\}.$$

Show that $W(x)$ is quasiconcave.

Q5. (15%) Show that $f(X) = (\mathbf{det} X)^{1/n}$ is concave on $\mathbf{S}^n_{++}$. (Hint: Use the same method of proof for showing the concavity of the log determinant function; see page 74 of the textbook)