ELEG5481 Signal Processing Optimization Techniques
Assignment 2

Feb. 18, 2013

Deadline: Feb. 27, 2013

Q1. (25%) Are the following sets convex? Give a brief justification for each of the following cases:
(a) $C = \{x \mid a_1^T x \geq b_1 \text{ or } a_2^T x \geq b_2\}$
(b) $C = \{x \in \mathbf{R}^n \mid \|x\|_2 = 1, \sum_{i=1}^n x_i = \sqrt{n}\}$
(c) $C = \{x | (a^T x + b)/(c^T x + d) \geq 1, \quad c^T x + d > 0\}$.
(d) $C = \{x | \max_{i=1,\dots,n} x_i \leq a\}$, where $a \in \mathbf{R}$.

Q2. (25%) Show that the following sets are convex:
(a) $C = \{x \in \mathbf{R}^n \mid \sum_{i=1}^n |x_i| \leq 1\}$.
(b) $C = \{x \mid \|x - c\| \leq a^T x + b\}$, where $\|\cdot\|$ is a norm.
(c) $C = \{x \mid B(x, a) \subset S\}$, where $a \geq 0$, S is convex set, and $B(x, a) = \{y \mid \|y - x\| \leq a\}$.
(d)

$$C = \{r \in \mathbf{R}^n \mid T(r) \succeq 0\}$$

where $T : \mathbf{R}^n \to \mathbf{S}^n$ is given by

$$T(r) = \begin{bmatrix} r_1 & r_2 & \dots & r_{n-1} & r_n \\ r_2 & r_1 & r_2 & & r_{n-1} \\ \vdots & r_2 & \ddots & \ddots & \vdots \\ r_{n-1} & & \ddots & \ddots & r_2 \\ r_n & r_{n-1} & & r_2 & r_1 \end{bmatrix}.$$

Q3. (25%) Let $x_0, \dots, x_k \in \mathbf{R}^n$. Consider the set of points that are closer (in Eluclidean norm) to x_0 than the other x_i, i.e.,

$$V = \{x \in \mathbf{R}^n \mid \|x - x_0\|_2 \leq \|x - x_i\|_2, \; i = 1, \dots, K\}.$$

V is called the Voronoi region around x_0 with respect to $x_1, \dots, x_K$.
(a) Show that V is a polyhedron, and thus convex. Express V in the form $V = \{x \mid Ax \preceq b\}$.
(b) Conversely, given a polyhedron P with nonempty interior, show how to find $x_0, \dots, x_K$ so that the polyhedron is the Voronoi region of x_0 with respect to $x_1, \dots, x_K$.

Q4. (25%) Let K be a cone. The set

$$K^* = \{y \mid x^T y \geq 0, \text{ for all } x \in K\}$$

is called the dual cone of K. Show that K^* is always a convex cone. Moreover, determine the dual cones of the following sets:
(a) $K = \mathbf{R}_+^n$.
(b) $K = \mathbf{S}_+^n$ (in which case the dual cone should be written as $K^* = \{Y \mid \mathbf{tr}(XY) \geq 0, \text{ for all } X \in K\}$).