ELEG5481 Signal Processing Optimization Techniques
Tutorial Solution 8

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Q1. Show that $X = B^T A^{-1} B$ solves the SDP

$$\begin{aligned} \min_{X \in \mathbf{S}^n} \quad & \mathbf{tr} X \\ \text{s.t.} \quad & \begin{bmatrix} A & B \\ B^T & X \end{bmatrix} \succeq 0, \end{aligned}$$

where $A \in \mathbf{S}^m_{++}$ and $B \in \mathbf{R}^{m \times n}$ are given data. Conclude that $\mathbf{tr}(B^T A^{-1} B)$ is a convex function of (A, B), in the domain of $\mathbf{S}^n_+ \times \mathbf{R}^{m \times n}$.

Solution: We will show that $X = B^T A^{-1} B$ is the unique solution indeed. By Schur's complement, we can rewrite the problem as

$$\begin{aligned} \min_{X \in \mathbf{S}^n} \quad & \mathbf{tr} X \\ \text{s.t.} \quad & X \succeq B^T A^{-1} B. \end{aligned}$$

Therefore, $X = B^T A^{-1} B$ is an feasible solution. We claim that $X = B^T A^{-1} B$ is the only point that satisfies the following two conditions

$$\begin{cases} \mathbf{tr} X \leq \mathbf{tr} B^T A^{-1} B \\ X \succeq B^T A^{-1} B. \end{cases}$$

Too see this, observe that $X_{ii} \geq [B^T A^{-1} B]_{ii}$ for $i = 1, \ldots, n$ and $\mathbf{tr} X = \sum_{i=1}^n X_{ii} \leq \mathbf{tr} B^T A^{-1} B$. Therefore, we have $X_{ii} = [B^T A^{-1} B]_{ii}$ and $[X - B^T A^{-1} B]_{ii} = 0$ for $i = 1, \ldots, n$. Because $X - B^T A^{-1} B \succeq 0$, we have $X = B^T A^{-1} B$. To conclude, no other feasible point produces a smaller objective value than $X = B^T A^{-1} B$. Therefore, $X = B^T A^{-1} B$ is the unique optimal solution.

For convenience, let $f(A, B) = \mathbf{tr}(B^T A^{-1} B)$. For any $A_1, A_2 \in \mathbf{S}^m_{++}$, $B_1, B_2 \in \mathbf{R}^{m \times n}$, and $0 \leq \theta \leq 1$, we have

$$\begin{aligned} f(\theta A_1 + (1-\theta) A_2, \theta B_1 + (1-\theta) B_2) = \min_{X \in \mathbf{S}^n} \quad & \mathbf{tr} X \\ \text{s.t.} \quad & \begin{bmatrix} \theta A_1 + (1-\theta) A_2 & \theta B_1 + (1-\theta) B_2 \\ \theta B_1^T + (1-\theta) B_2^T & X \end{bmatrix} \succeq 0. \end{aligned}$$

We can replace X by $\theta X_1 + (1-\theta) X_2$ where $X_1, X_2 \in \mathbf{S}^n_+$. Therefore,

$$\begin{aligned} f(\theta A_1 + (1-\theta) A_2, \theta B_1 + (1-\theta) B_2) = \min_{X_1, X_2 \in \mathbf{S}^n} \quad & \mathbf{tr} \theta X_1 + (1-\theta) X_2 \\ \text{s.t.} \quad & \begin{bmatrix} \theta A_1 + (1-\theta) A_2 & \theta B_1 + (1-\theta) B_2 \\ \theta B_1^T + (1-\theta) B_2^T & \theta X_1 + (1-\theta) X_2 \end{bmatrix} \succeq 0. \end{aligned}$$

which can be upper bounded by

$$\begin{aligned} \min_{X_1, X_2 \in \mathbf{S}^n} \quad & \mathbf{tr} \theta X_1 + (1-\theta) X_2 \\ \text{s.t.} \quad & \begin{bmatrix} A_1 & B_1 \\ B_1^T & X_1 \end{bmatrix} \succeq 0, \\ & \begin{bmatrix} A_2 & B_2 \\ B_2^T & X_2 \end{bmatrix} \succeq 0. \end{aligned}$$

This is just $\theta f(A_1, B_1) + (1-\theta) f(A_2, B_2)$.

Q2. Formulate the following complex-valued infinity norm minimization problem as an SOCP.

$$\begin{aligned} \min_{x \in \mathbf{C}^n} \quad & \|x\|_\infty \\ \text{s.t.} \quad & Ax = b, \end{aligned}$$

where $A \in \mathbf{C}^{m \times n}$ and $b \in \mathbf{C}^m$. Hint: transform the complex-valued data and variable to real-valued data and variable.

Solution: Denote $x = x_R + \boldsymbol{j} x_I$, where $x_R \in \mathbf{R}^n$ and $x_I \in \mathbf{R}^n$ are the real and imaginary parts of x, respectively. Similarly, let $A = A_R + \boldsymbol{j} A_I$ and $b = b_R + \boldsymbol{j} b_I$. Then we can rewrite the problem as

$$\begin{aligned} \min_{x_R, x_I \in \mathbf{R}^n} \quad & \max_{i=1,\ldots,n} \sqrt{x_{R,i}^2 + x_{I,i}^2} \\ \text{s.t.} \quad & \begin{bmatrix} A_R & -A_I \\ A_I & A_R \end{bmatrix} \begin{bmatrix} x_R \\ x_I \end{bmatrix} = \begin{bmatrix} b_R \\ b_I \end{bmatrix}, \end{aligned}$$

Adding a slack variable t, we have

$$\begin{aligned} \min_{x_R, x_I \in \mathbf{R}^n} \quad & t \\ \text{s.t.} \quad & \sqrt{x_{R,i}^2 + x_{I,i}^2} \le t, \quad i = 1, \ldots, n, \\ & \begin{bmatrix} A_R & -A_I \\ A_I & A_R \end{bmatrix} \begin{bmatrix} x_R \\ x_I \end{bmatrix} = \begin{bmatrix} b_R \\ b_I \end{bmatrix}, \end{aligned}$$

which is an SOCP.

Q3. Consider the problem, with variable $x \in \mathbf{R}^n$,

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & c^T x \\ \text{s.t.} \quad & Ax \preceq b \text{ for all } A \in \mathcal{A}, \end{aligned}$$

where $\mathcal{A} \subset \mathbf{R}^{m \times n}$ is the set

$$\mathcal{A} = \{A \in \mathbf{R}^{m \times n} \mid \bar{A}_{ij} - V_{ij} \le A_{ij} \le \bar{A}_{ij} + V_{ij}, i = 1, \ldots, m, j = 1, \ldots, n\}.$$

(The matrices $\bar{A}$ and V are given.) Express this problem as an LP.

Solution: Let us first characterize the feasible set of x. Consider the following problem

$$\begin{aligned} \max_{a_i \in \mathbf{R}^n} \quad & a_i^T x \\ \text{s.t.} \quad & a_i \in \mathcal{A}_i, \end{aligned}$$

where $\mathcal{A}_i = \{a_i \in \mathbf{R}^n \mid \bar{A}_{ij} - V_{ij} \le a_{ij} \le \bar{A}_{ij} + V_{ij}, j = 1, \ldots, n\}$. Obviously, an optimal solution is given by some $a_{ij} = \bar{A}_{ij} \pm V_{ij}$ for $i = 1, \ldots, m$. Therefore, the constraint $Ax \preceq b$ for all $A \in \mathcal{A}$ can be written as for $i = 1, \ldots, m$,

$$\sum_{j=1}^{n} (A_{ij} \pm V_{ij}) x_j \le b, \text{ for all possible choice of sign of } V_{ij}.$$

This again can be written as

$$Ax + V|x| \le b$$

Therefore, the original problem can be written as

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & c^T x \\ \text{s.t.} \quad & Ax + V|x| \le b, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \min_{x, y \in \mathbf{R}^n} \quad & c^T x \\ \text{s.t.} \quad & Ax + Vy \le b, \\ & -y \le x \le y. \end{aligned}$$