ELEG5481 Signal Processing Optimization Techniques
Tutorial 5

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Q1. Please point out what are the flaws in the following arguments.

(a) Question: Is the set $C = \{x \mid a_1^T x \leq b_1 \text{ or } a_2^T x \leq b_2\}$ convex?
No. The union of two convex is not convex.

(b) Question: Show that the set $V = \{x \in \mathbf{R}^n \mid \|x - x_0\| \leq \|x - x_i\|, i = 1, \ldots, n\}$ is a polyhedron.
Argument: Because $\|x - x_0\| \leq \|x - x_i\| \Longrightarrow (x_0 - x_i)^T x \geq \frac{1}{2}(x_0^T x_0 - x_i^T x_i)$ for $i = 1, \ldots, n$, we have

$$V = \{x \mid (x_0 - x_i)^T x \geq \tfrac{1}{2}(x_0^T x_0 - x_i^T x_i), i = 1, \ldots, n\}.$$

Therefore V is a polyhedron.

(c) Given a polyhedron $P = \{x \mid Ax \preceq b\}$ with nonempty interior, show how to find $x_0, \ldots, x_K$ such that P is $V(x_0)$.
Argument: Let x_i be a point in the interior of P. Denote the ith row of A as a_i^T. Let x_i be the mirror image of x_0 with respect to the hyperplane $a_i^T x = b$.

(d) Question: Show the following set is convex: $C = \{x \mid B(x, a) \subset S\}$, where $a \geq 0$, S is a convex set, and $B(x, a) = \{y \mid \|y - x\| \leq a\}$.
Argument: Let $x_1, x_2 \in C$, and $0 \leq \theta \leq 1$. We have $\|y - (\theta x_1 + (1 - \theta) x_2)\| \leq a$. (The remaining argument is skipped.)

Q2. Prove the Hadamard's inequality

$$\mathbf{det} P \leq \prod_{i=1}^{n} P_{ii},$$

where $P \in \mathbf{S}_+^n$.

(a) Step1. (*Cholesky decomposition*) Show that a PD matrix P can be decomposed as $P = R^T R$, where R is an upper triangular matrix with positive diagonal entries.

(b) Step2. Use the Cholesky decomposition to prove that the Hadamard's inequality is true for a PD matrix P.

(c) Step3. Show that the Hadamard's inequality is true for a non-PD symetric matrix P.

Q3. Show that

$$f(x) = \prod_{k=1}^{n} x_k^{\alpha_k}, \quad \mathbf{dom}\, f = \mathbf{R}_{++}^n,$$

is concave, where α_k are nonegative numbers with $\sum_k \alpha_k = 1$. Hint:

(a) Step1. Show the following result first: For a symmetric matrix $A \in \mathbf{R}^{n \times n}$, if A is diagonally dominant (i.e. $|A_{i,i}| \geq \sum_{j \neq i} |A_{j,i}|$, for $i = 1, \ldots, n$) and if $A_{i,i} > 0$ for $i = 1, \ldots, n$, then A is positive semidefinite.

(b) Step2. Compute the Hessian of $f(x)$ and show that it is negative semidefinite.