ELEG5481 Signal Processing Optimization Techniques
Tutorial 3

Feb. 18, 2013

Review on Convexity preserving operations and generalized inequality

Convexity preserving operations:

- Intersection of convex sets: $S = \bigcap_{\alpha \in \mathcal{A}} S_\alpha$ is convex if S_α is convex, where $\mathcal{A}$ is an arbitrary index sets.
- Image under affine mapping: Let $f(x) = Ax + b$ be an affine function. Then $\{f(x) \mid x \in C\}$ is convex if C is convex.
- Inverse image under affine mapping: Let $f(x) = Ax+b$ be an affine function. $\{x \mid f(x) \in C\}$ is convex if C is convex.

A convex cone K is a proper cone if

- K is closed (the boundary of K is in K)
- K is solid (has nonempty interior)
- K is pointed ($x \in K$ and $x \in -K$ imply $x = 0$).

Example:

- Nonnegative orthant $K = \mathbf{R}^n_+$
- SOC $K = \{(x, t) \mid \|x\|_2 \leq t\}$
- PSD cone $K = \{X \in \mathbf{S}^+ \mid X \succeq \mathbf{0}\}$

Generalized inequality defined by a proper cone K:

$$x \preceq_K y \Longleftrightarrow y - x \in K$$
$$x \prec_K y \Longleftrightarrow y - x \in \text{int}K$$

We say that $x \in S$ is the minimum element of S if for any element $y \in S$ we have $x \preceq_K y$. Loosely speaking, this means that x is the smallest element in S.

We say that $x \in S$ is a minimal element of S if for any element $y \in S$ we have $y \preceq_K x$ only when $y = x$. Loosely speaking, this means that nobody in S (except x itself) is smaller than or equal to x.

Note that we may have neither $x \preceq_K y$ nor $y \preceq_K x$, i.e. x is not comparable to y. Hence nobody in S smaller than or equal to x does not means that x is the smallest.

Q1. *Cone of polynomials nonngegative on [0,1].* Let K be defined as

$$K = \{c \in \mathbf{R}^n \mid c_1 + c_2 t + \ldots + c_n t^{n-1} \geq 0 \text{ for } t \in [0,1]\},$$

i.e., K is the cone of (coefficients of) polynomials of degree $n-1$ that are nonnegative on the interval $[0,1]$. Show that K is a proper cone.

Q2. A set C in $\mathbf{R}^m$ is convex if and only if every convex combination of vectors from C again is a vector from C, i.e. $x = \sum_{i=1}^n \lambda_i x_i$ is in C, where $\sum_{i=1}^n \lambda_i = 1$, $\lambda_i \geq 0$, and $x_i \in C$.

Q3. Show that the convex hull of a set S is the intersection of all convex sets that contain S.

Q4. What are the interiors the following sets
(a) $C = \{x \mid a^T x = b\}$, where $a \neq 0$.
(b) $C = \{x \mid a^T x \leq b\}$, where $a \neq 0$.
(c) $C = \{x \mid x^T x = 1\}$.
(d) $C = \{x \mid x^T x \leq 1\}$.