ELEG5481 Signal Processing Optimization Techniques
Tutorial 1

Jan. 24, 2013

Q1. Let x and y be two vector in $\mathbf{R}^n$, and $y \neq 0$. Show that x can be decomposed uniquely in the form of $x = x_\perp + x_\|$, where $x_\| = cy$ for some c, and $x_\perp^T y = 0$. Moreover, $\|x_\|\|_2^2 + \|x_\perp\|_2^2 = \|x\|_2^2$.

Q2. Prove the following inequalities:

(a) Cauchy-Schwartz inequality:
$$|x^T y| \leq \|x\|_2 \|y\|_2,$$
where the equality holds if and only if $x = cy$ or $y = cx$ for some c.

(b) Hölder inequality:
$$|x^T y| \leq \|x\|_p \|y\|_q,$$
where $1/p + 1/q = 1$, $p \geq 1$ and $q \geq 1$.

Q3. Prove the following functions are norms:
(a) $f(x) = \|x\|_\infty$.

(b) $f(z) = \|z\|_* = \sup\{z^T x \mid \|x\| \leq 1\}$ where $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

Q4. Prove the function $f(x) = \|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ with $0 < p < 1$ is not a norm.

Q5. Prove the following statements
(a) If $\mathcal{S}$ is a nonempty subspace in $\mathbf{R}^n$, then $\mathcal{S} = \mathcal{R}(A)$ for some matrix A.

(b) For any $A \in \mathbf{R}^{m\times n}$, $\mathcal{N}(A) = \mathcal{R}(A^T)^\perp$, where $\mathcal{R}(A^T)^\perp$ is the orthogonal complement of the range space spanned by A^T.