ELEG5481 Signal Processing Optimization Techniques
Tutorial Solution 6

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Q1. Consider the following problem.

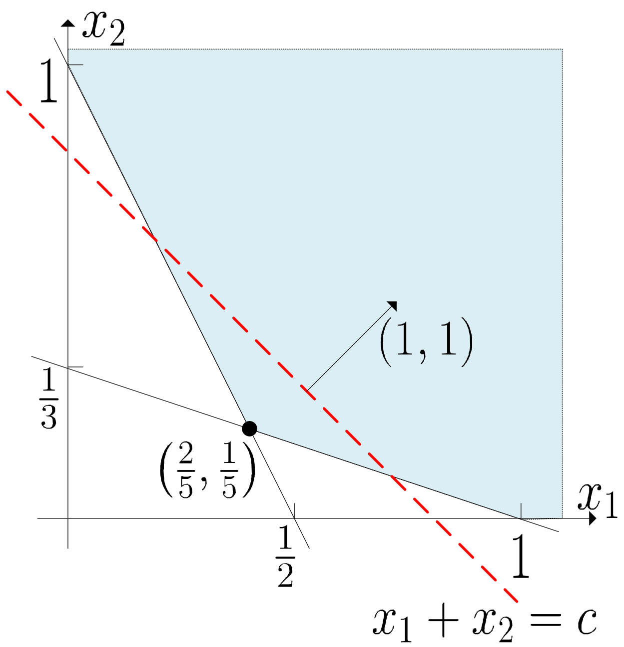
$$\begin{aligned} \min_{x_1,x_2} \quad & f(x_1,x_2) = x_1 + x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

(a) Sketch the feasible set and find the optimal solution.

(b) Use the optimality condition to verify the obtained solution is optimal.

Solution:

(a) The blue region in the following figure is the feasible set. The red line is $x_1 + x_2 = c$. It can be seen that to minimize the objective function, we need to find the smallest c such that the red line can intersect the feasible set. The minimum c can be found when the red line intersects the point $(\frac{1}{5}, \frac{2}{5})$.



(b) To verify that $x^\star = (\frac{2}{5}, \frac{1}{5})$ is an optimal solution, we need to show that

$$\nabla f^T(x^\star)(x - x^\star) \geq 0 \text{ for all feasible } x.$$

Indeed, for any feasible x, we have

$$\nabla f(x^\star)^T(x - x^\star) = (x_1 - \tfrac{2}{5}) + (x_2 - \tfrac{1}{5}) = \tfrac{2}{5}(2x_1 + x_2) + \tfrac{1}{5}(x_1 + 3x_2) - \tfrac{3}{5} \geq 0.$$

Q2. Consider the following three problems, where $a_i \in \mathbf{R}^n, i = 1, \ldots, m$, $b \in \mathbf{R}^m$, and $M > 0$ are given data.

1. The robust least-squares problem

$$\min_{x} \quad \sum_{i=1}^{m} \phi(a_i^T x - b_i),$$

where $x \in \mathbf{R}^n$, and $\phi : \mathbf{R} \to \mathbf{R}$ is defined as

$$\phi(u) = \begin{cases} u^2, & |u| \le M, \\ M(2|u| - M), & |u| > M. \end{cases}$$

2. The least-squares problem with variable weights

$$\begin{aligned} \min_{x,w} \quad & \sum_{i=1}^{m} f(a_i^T x - b_i, w_i) \\ \text{s.t.} \quad & w \ge 0, \end{aligned}$$

where $x \in \mathbf{R}^n$, and $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is defined as

$$f(u, w) = u^2/(w + 1) + M^2 w.$$

3. The quadratic program

$$\min_{x} \quad \sum_{i=1}^{m} g(a_i^T x - b)$$

where $g(u)$ is defined as

$$\begin{aligned} g(u) = \min_{s,t} \quad & s^2 + 2Mt \\ \text{s.t.} \quad & |u| \le s + t, \\ & 0 \le s \le M, \\ & t \ge 0. \end{aligned}$$

Show the following results.
(a) Show that problem (1) and problem (2) are equivalent by showing that

$$\phi(u) = \min_{w \ge 0} f(u, w).$$

(b) Show that problem (1) and problem (3) are equivalent by showing that

$$\phi(u) = g(u).$$

Solution:
(a) To find the minimum of $f(u, w)$ over $w \ge 0$, we compute the gradient of $f(u, w)$ with respect to w. Indeed, we have

$$\frac{\partial f}{\partial w} = -\frac{u^2}{(w + 1)^2} + M^2.$$

We can see that if $|u| > M$, then $f(u, w)$ decreases in the region $w \in [0, \frac{|u|}{M} - 1]$ and increases in the region $(\frac{|u|}{M} - 1, \infty)$. Therefore, $f(u, w)$ attains the minimum $M(2|u| - M)$ at $w = \frac{|u|}{M} - 1$. On the other hand, if $|u| \le M$, then $f(u, w)$ increases in the region $w \in [0, \infty)$. Therefore, $f(u, w)$ attains minimum u^2 at $w = 0$.

(b) First observe that at optimum, s and t must satisfies $|u| = s + t$. Otherwise, we can reduce s or t a little so that feasibility is maintained and the objective value is reduced. Therefore substituting $t = |u| - s$ back, we can eliminate t and the problem is equivalent to

$$\begin{aligned} g(u) = \min_{s,t} \quad & s^2 + 2M(|u| - s) \\ \text{s.t.} \quad & 0 \leq s \leq M, \\ & s \leq |u|. \end{aligned}$$

Note that the objective function is quadratic, and attains the minimum at $s = M$ if the problem is unconstrained. Therefore, when $M < |u|$, $g(u) = M^2 + 2M(|u| - M) = M(2|u| - M)$. When $M \geq |u|$, $g(u) = |u|^2$.