# ELEG5481 Signal Processing Optimization Techniques
# Tutorial Solution 4

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**Q1.** *Cone of polynomials nonngegative on [0,1].* Let $K$ be defined as

$$K = \{c \in \mathbf{R}^n \ | \ c_1 + c_2 t + \ldots + c_n t^{n-1} \geq 0 \text{ for } t \in [0,1]\},$$

i.e., $K$ is the cone of (coefficients of) polynomials of degree $n-1$ that are nonnegative on the interval $[0,1]$. Show that $K$ is a proper cone.

**Solution:** For convenience we define $f_c(t) = c_1 + c_2 t + \ldots + c_n t^{n-1}$.

*$K$ is a cone.* If $c \in K$, then $f_c(t) \geq 0$ for $t \in [0,1]$, which implies $f_{ac}(t) \geq 0$ for $t \in [0,1]$ for all $a \geq 0$. Therefor $ac$ belongs to $K$.

*$K$ is convex.* Suppose $c$ and $c'$ belong to $K$, and $0 \leq \lambda \leq 1$. Then we have

$$f_{\lambda c + (1-\lambda)c'}(t) = f_{\lambda c}(t) + f_{(1-\lambda)c'}(t)$$

As $f_c(t)$ and $f_{c'}(t)$ are nonnegative on $t \in [0,1]$, so are $f_{\lambda c}(t)$ and $f_{(1-\lambda)c'}(t)$. Therefore $f_{\lambda c+(1-\lambda)c'}(t)$ is nonnegative on $t \in [0,1]$ as well.

We can also consider $K$ as the intersection of half spaces, as we can write $K$ as

$$K = \bigcap_{t \in [0,1]} \{c \in \mathbf{R}^n | f_c(t) \geq 0\}.$$

For a given $t$, $\{c \in \mathbf{R}^n | f_c(t) \geq 0\}$ is just a half space. Intersection of closed sets is closed.

*$K$ is pointed.* We need to show that $f_c(t) \geq 0$ and $f_{-c}(t) \geq 0$ on $t \in [0,1]$ implies $c = 0$. To show that $c_1 = 0$, observe that $f_c(0) = c_1 \geq 0$ and $f_{-c}(0) = -c_1 \geq 0$. Therefore $c_1 = 0$. Thus, we can write $f_c(t)$ in the form of $f_c(t) = t f_{c,1}(t)$, where

$$f_{c,1}(t) = c_2 + c_3 t + \ldots + c_n t^{n-2}.$$

Since $f_c(t) \geq 0$ on $t \in [0,1]$, we have $f_{c,1}(t) \geq 0$ on $t \in [0,1]$ as well. Thus $f_{c,1}(0) = c_2 \geq 0$. Similarly, we have $c_2 \leq 0$. Therefore $c_2 = 0$. Continuing this process, we conclude that $c = 0$.

*$K$ has nonempty interior.* We show that the all-one vector 1 lies in the interior of $K$. We show this by showing that for $c$ with $\|c - 1\| \leq 1$ belongs to $K$, i.e. $f_c(t) \geq 0$ on $t \in [0,1]$. As $\|c - 1\| \leq 1$, we have $c \succeq 0$, therefore $f_c(t) \geq 0$ on $t \in [0,1]$.

*$K$ is closed.* We can write $K$ as

$$K = \bigcap_{t \in [0,1]} \{c \in \mathbf{R}^n | f_c(t) \geq 0\}.$$

For a given $t$, $\{c \in \mathbf{R}^n | f_c(t) \geq 0\}$ is just a closed half space. By the result from mathematical analysis that arbitrary intersection of closed sets is closed, we conclude that $K$ is closed.

**Q2.** Show by definition that the function $f(x) = \|Ax - b\|$ is convex.

**Solution:**

$$f(\theta x + (1-\theta)y) = \|\theta(Ax - b) + (1-\theta)(Ay - b)\| \leq \theta\|Ax - b\| + (1-\theta)\|Ay - b\| = \theta f(x) + (1-\theta) f(y)$$

**Q3.** Show by the first order condition that the function $f(x) = 1/(x_1x_2)$ with domain $\mathbf{R}^2_{++}$ is convex.

**Solution:** We have

$$\nabla f = -\begin{bmatrix} \frac{1}{x_1^2x_2} \\ \frac{1}{x_1x_2^2} \end{bmatrix}.$$

We need to show that $f(y) \geq f(x) + \nabla f(x)^T(y-x)$ is true for all $x$ and $y$. Indeed we have

$$f(y) - f(x) - \nabla f(x)^T(y-x) = \left(\frac{1}{y_1y_2} + \frac{y_1}{x_1^2x_2} + \frac{y_2}{x_1x_2^2}\right) - \frac{3}{x_1x_2} \geq 0$$

where the inequality is due to the arithmetic mean- geometric mean inequality for $x \in \mathbf{R}^n_+$,

$$\frac{\sum_{i=1}^n x_i}{n} \geq \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}}$$

**Q4.** Show by using the second order condition that the function $f(x,t) = -\log(t^2 - x^Tx)$ is convex in the domain $\{(x,t) \in \mathbf{R}^n \times \mathbf{R} \mid t > \|x\|_2\}$.

**Solution:**

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= \frac{2x_i}{t^2 - x^Tx} \\ \frac{\partial f}{\partial t} &= -\frac{2t}{t^2 - x^Tx}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial x_i^2} &= \frac{2}{(t^2 - x^Tx)^2}(2x_i^2 + t^2 - x^Tx) \\ \frac{\partial^2 f}{\partial x_i \partial x_j} &= \frac{2}{(t^2 - x^Tx)^2}(2x_ix_j) \\ \frac{\partial^2 f}{\partial t \partial x_i} &= \frac{2}{(t^2 - x^Tx)^2}(-2x_it) \\ \frac{\partial^2 f}{\partial t^2} &= \frac{2}{(t^2 - x^Tx)^2}(t^2 + x^Tx) \end{aligned}$$

Therefore, we have

$$\begin{aligned} \nabla^2 f(x,t) =& \frac{2}{(t^2 - x^Tx)^2}\begin{bmatrix} 2xx^T + (t^2 - x^Tx)I & -2tx \\ -2tx^T & t^2 + x^Tx \end{bmatrix} \\ =& \frac{2}{(t^2 - x^Tx)^2}\left(2\begin{bmatrix} xx^T & -tx \\ -tx^T & t^2 \end{bmatrix} + \begin{bmatrix} (t^2 - x^Tx)I & \\ & -(t^2 - x^Tx) \end{bmatrix}\right). \end{aligned}$$

We now verify that $\nabla^2 f(x,t)$ is PSD. Let $(y,s) \in \mathbf{R}^n \times \mathbf{R}$. Then,

$$\frac{(t^2 - x^Tx)^2}{2}(y^T, s)\nabla^2 f(x,t)(y^T, s)^T = 2(ts - x^Ty)^2 - (t^2 - x^Tx)(s^2 - y^Ty) \tag{1}$$

For $(y,s)$ that satisfies $s^2 \leq y^Ty$, (1) is greater than zero. Next, let us assume that $s^2 > y^Ty$. We have

$$|ts - x^Ty| \geq |ts| - |x^Ty| \geq |ts| - \|x\|\|y\| \geq 0$$

Therefore

$$(1) \geq 2(|ts| - \|x\|\|y\|)^2 - (t^2 - x^Tx)(s^2 - y^Ty) = (|ts| - \|x\|\|y\|)^2 + (|t|\|y\| - |s|\|x\|)^2 \geq 0.$$