ELEG5481 Signal Processing Optimization Techniques
Tutorial Solution 3

Feb. 18, 2013

Review on Convexity preserving operations and generalized inequality

Convexity preserving operations:

- Intersection of convex sets: $S = \bigcap_{\alpha \in \mathcal{A}} S_\alpha$ is convex if S_α is convex, where $\mathcal{A}$ is an arbitrary index sets.
- Image under affine mapping: Let $f(x) = Ax + b$ be an affine function. Then $\{f(x) \mid x \in C\}$ is convex if C is convex.
- Inverse image under affine mapping: Let $f(x) = Ax+b$ be an affine function. $\{x \mid f(x) \in C\}$ is convex if C is convex.

A convex cone K is a proper cone if

- K is closed (the boundary of K is in K)
- K is solid (has nonempty interior)
- K is pointed ($x \in K$ and $x \in -K$ imply $x = 0$).

Example:

- Nonnegative orthant $K = \mathbf{R}^n_+$
- SOC $K = \{(x,t) \mid \|x\|_2 \leq t\}$
- PSD cone $K = \{X \in \mathbf{S}^+ \mid X \succeq \mathbf{0}\}$

Generalized inequality defined by a proper cone K:

$$x \preceq_K y \Longleftrightarrow y - x \in K$$
$$x \prec_K y \Longleftrightarrow y - x \in \text{int}K$$

We say that $x \in S$ is the minimum element of S if for any element $y \in S$ we have $x \preceq_K y$. Loosely speaking, this means that x is the smallest element in S.

We say that $x \in S$ is a minimal element of S if for any element $y \in S$ we have $y \preceq_K x$ only when $y = x$. Loosely speaking, this means that nobody in S (except x itself) is smaller than or equal to x.

Note that we may have neither $x \preceq_K y$ nor $y \preceq_K x$, i.e. x is not comparable to y. Hence nobody in S smaller than or equal to x does not means that x is the smallest.

Q1. *Cone of polynomials nonngegative on [0,1].* Let K be defined as

$$K = \{c \in \mathbf{R}^n \mid c_1 + c_2 t + \ldots + c_n t^{n-1} \geq 0 \text{ for } t \in [0,1]\},$$

i.e., K is the cone of (coefficients of) polynomials of degree $n-1$ that are nonnegative on the interval $[0,1]$. Show that K is a proper cone.

Solution: For convenience we define $f_c(t) = c_1 + c_2 t + \ldots + c_n t^{n-1}$.

K is a cone. If $c \in K$, then $f_c(t) \geq 0$ for $t \in [0,1]$, which implies $f_{ac}(t) \geq 0$ for $t \in [0,1]$ for all $a \geq 0$. Therefor ac belongs to K.

K is convex. Suppose c and c' belong to K, and $0 \leq \lambda \leq 1$. Then we have

$$f_{\lambda c + (1-\lambda)c'}(t) = f_{\lambda c}(t) + f_{(1-\lambda)c'}(t)$$

As $f_c(t)$ and $f_{c'}(t)$ are nonnegative on $t \in [0,1]$, so are $f_{\lambda c}(t)$ and $f_{(1-\lambda)c'}(t)$. Therefore $f_{\lambda c + (1-\lambda)c'}(t)$ is nonnegative on $t \in [0,1]$ as well.

We can also consider K as the intersection of half spaces, as we can write K as

$$K = \bigcap_{t \in [0,1]} \{c \in \mathbf{R}^n | f_c(t) \geq 0\}.$$

For a given t, $\{c \in \mathbf{R}^n | f_c(t) \geq 0\}$ is just a half space. Intersection of closed sets is closed.

K is pointed. We need to show that $f_c(t) \geq 0$ and $f_{-c}(t) \geq 0$ on $t \in [0,1]$ implies $c = 0$. To show that $c_1 = 0$, observe that $f_c(0) = c_1 \geq 0$ and $f_{-c}(0) = -c_1 \geq 0$. Therefore $c_1 = 0$. Thus, we can write $f_c(t)$ in the form of $f_c(t) = t f_{c,1}(t)$, where

$$f_{c,1}(t) = c_2 + c_3 t + \ldots + c_n t^{n-2}.$$

Since $f_c(t) \geq 0$ on $t \in [0,1]$, we have $f_{c,1}(t) \geq 0$ on $t \in [0,1]$ as well. Thus $f_{c,1}(0) = c_2 \geq 0$. Similarly, we have $c_2 \leq 0$. Therefore $c_2 = 0$. Continuing this process, we conclude that $c = 0$.

K has nonempty interior. We show that the all-one vector $\mathbf{1}$ lies in the interior of K. We show this by showing that for c with $\|c - \mathbf{1}\| \leq 1$ belongs to K, i.e. $f_c(t) \geq 0$ on $t \in [0,1]$. As $\|c - \mathbf{1}\| \leq 1$, we have $c \succeq 0$, therefore $f_c(t) \geq 0$ on $t \in [0,1]$.

K is closed. We can write K as

$$K = \bigcap_{t \in [0,1]} \{c \in \mathbf{R}^n | f_c(t) \geq 0\}.$$

For a given t, $\{c \in \mathbf{R}^n | f_c(t) \geq 0\}$ is just a closed half space. By the result from mathematical analysis that arbitrary intersection of closed sets is closed, we conclude that K is closed.

Q2. A set C in $\mathbf{R}^m$ is convex if and only if every convex combination of vectors from C again is a vector from C, i.e. $x = \sum_{i=1}^n \lambda_i x_i$ is in C, where $\sum_{i=1}^n \lambda_i = 1$, $\lambda_i \geq 0$, and $x_i \in C$.

Solution: If every convex combination of vectors from C is a vector from C, then the convex combination of two vectors from C is a vector from C. This is just the definition of convex set.

Conversely, we need to show that if C is convex then $x = \sum_{i=1}^n \lambda_i x_i$ is in C, where $\sum_{i=1}^n \lambda_i = 1$, $\lambda_i \geq 0$, and $x_i \in C$. We show this by induction. For $n = 2$, this is obviously true as this is the definition of convex set. Assuming this is true for $n = k$, we need to show that it is true for $n = k+1$. For $n = k+1$, x can be written as

$$x = \lambda_{k+1} x_{k+1} + (\sum_{i=1}^{k} \lambda_i x_i).$$

If $\lambda_{k+1} = 0$, then x is a convex combination of k points, and thus belongs to C. If $\lambda_{k+1} = 1$, then $x = x_{k+1}$ belongs to C. If $0 < \lambda_{k+1} < 1$, we have

$$x = \lambda_{k+1} x_{k+1} + (1 - \lambda_{k+1})\tilde{x},$$

where

$$\tilde{x} = \sum_{i=1}^{k} \frac{\lambda_i}{(1 - \lambda_{k+1})} x_i.$$

Note that $\sum_{i=1}^{k} \frac{\lambda_i}{(1-\lambda_{k+1})} = 1$, hence $\tilde{x}$ belongs to C. As x is a convex combination of x_{k+1} and $\tilde{x}$, x belongs to C.

Q3. Show that the convex hull of a set S is the intersection of all convex sets that contain S.

Solution: Let D denote a convex set that contains S. Then the intersection of all convex sets that contain S can be denoted by

$$C = \bigcap_{\substack{D \text{ convex} \\ S \subset D}} D.$$

Then for any $x \in \text{co}S$, x is a convex combination of some points $\{x_i\}_{i=1}^n$ in $\mathcal{S}$. Then $\{x_i\}_{i=1}^n$ belongs to $\mathcal{S}$, and thus D. Because D is convex and x is a convex combination of $\{x_i\}_{i=1}^n$ that belong to D, x belongs to D.

Conversely, since co$S$ is convex and contain S, by definition of C, C is a subset of co$S$.

Q4. What are the interiors of the following sets

(a) $C = \{x \mid a^T x = b\}$, where $a \neq 0$.
(b) $C = \{x \mid a^T x \leq b\}$, where $a \neq 0$.
(c) $C = \{x \mid x^T x = 1\}$.
(d) $C = \{x \mid x^T x \leq 1\}$.

Solution:

(a) int$C = \emptyset$.
(b) int$C = \{x \mid a^T x < b\}$
(c) int$C = \emptyset$.
(d) int$C = \{x \mid x^T x < 1\}$.