ELEG5481 Signal Processing Optimization Techniques
Tutorial Solution 2

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Q1. Let $A = U\Sigma V^H$ be a SVD of matrix $A \in \mathbf{C}^{n\times n}$. Show that the problem $\max\{\Re\{\text{tr}AW\} \mid W \in \mathbf{C}^{n\times n} \text{ is unitary}\}$ has the solution $W = VU^H$, and the value of the maximum is $\sum_{i=1}^n \sigma_i$.

Solution: Let $Q = V^H W U$. Then Q is unitary i.f.f W is unitary. We have that

$$\max\{\Re\{\text{tr}AW\} \mid W \in \mathbf{C}^{n\times n} \text{ is unitary}\}$$

is the same as

$$\max\{\Re\{\text{tr}\Sigma Q\} \mid Q \in \mathbf{C}^{n\times n} \text{ is unitary}\}.$$

The objective function can be written as $\sum_{i=1}^n \sigma_i \Re\{Q_{ii}\}$. Since Q is unitary, we have $|Q_{ii}| \leq 1$. Therefore

$$\max_{Q \text{ is unitary}} \left\{ \sum_{i=1}^n \sigma_i \Re\{Q_{ii}\} \right\} \leq \sum_{i=1}^n \sigma_i.$$

Equality holds if $Q_{ii} = 1$ for $i = 1, \ldots, n$. This can be achieved by $Q = I$. Hence $W = VU^H$ is a solution of the original problem.

Q2. If A is Hermitian, show that there exists a solution $x^\star$ that is optimal to the following two the optimization problems

(a) $v_1 = \max_{x^H x = 1} f_1(x) = x^H A x$.

(b) $v_2 = \max_{x \neq 0} f_2(x) = \frac{x^H A x}{x^H x}$.

Solution: If we show that $v_1 = v_2$, then an optimal solution $x^\star$ of the first problem satisfy $v_1 = f_1(x^\star)$, and $f(x^\star) \leq v_2$. But $f_1(x^\star) = f_2(x^\star)$. This implies that $x^\star$ is also optimal to the second problem. We now show that $v_1 = v_2$. Obviously $v_1 \leq v_2$, because

$$v_2 = \max_{x\neq 0} \frac{x^H A x}{x^H x} \geq \max_{x \neq 0, x^H x = 1} \frac{x^H A x}{x^H x} = \max_{x^H x = 1} \frac{x^H A x}{x^H x} = \max_{x^H x = 1} x^H A x = v_1.$$

We also have $v_2 \leq v_1$, since for all $x \neq 0$, we have $f_2(x) = f_1(x/\|x\|_2) \leq v_1$.

Q3. Prove that for $p \geq 2$,

$$\|x\|_2 \leq n^{-\frac{2-p}{2p}} \|x\|_p,$$

by showing that

(a) The inequality above is true if the following equation is true

$$n^{\frac{2-p}{2}} = \min_{\substack{\|x\|_2 = 1 \\ x_j \geq 0, j = 1, \ldots, n}} \|x\|_p^p. \tag{1}$$

(b) Show that (1) is true for $n = 2$.

(c) Assuming that (1) is true for $n = i - 1$, show that (1) is true for $n = i$.

Solution:

(a) We prove that the inequality is true if the following equation is true

$$n^{\frac{2-p}{2p}} = \min_{\|x\|_2=1} \|x\|_p. \tag{2}$$

If this equation is true, then

$$n^{\frac{2-p}{2p}} \|x\|_2 \le \|x\|_p, \quad \forall\ x \text{ with } \|x\|_2 = 1.$$

This implies

$$n^{\frac{2-p}{2p}} \|x\|_2 \le \|x\|_p, \quad \forall\ x.$$

The reason is that for $x \neq 0$, we have

$$n^{\frac{2-p}{2p}} \left\| \frac{x}{\|x\|_2} \right\|_2 \le \left\| \frac{x}{\|x_2\|} \right\|_p$$

(b) For $n = 2$, (1) can be written as

$$\begin{aligned} \min \quad & x_1^p + x_2^p \\ \text{s.t.} \quad & x_1^2 + x_2^2 = 1. \end{aligned}$$

which can be simplified as

$$\begin{aligned} \min \quad & f_2(x_2) = x_2^p + (1 - x_2^2)^{\frac{p}{2}} \\ \text{s.t.} \quad & 0 \le x_2 \le 1. \end{aligned}$$

This problem is to find the minimum value of $f_2(x_2)$ in the range of $0 \le x_2 \le 1$. This can be done by observing the derivative

$$f_2'(x_2) = px_2(x_2^{p-2} - (1 - x_2^2)^{\frac{p}{2}-1})$$

It can be seen that

$$f_2'(x_2) \begin{cases} < 0, & 0 \le x_2 < \frac{1}{\sqrt{2}}, \\ = 0, & x_1 = \frac{1}{\sqrt{2}}, \\ > 0, & \frac{1}{\sqrt{2}} < x_2 \le 1. \end{cases}$$

This means that $f_2(x_2)$ is decreasing in the range of $0 \le x_1 \le \frac{1}{\sqrt{2}}$, and is increasing in the range $\frac{1}{\sqrt{2}} \le x_2 \le 1$. Therefore the minimum value of $f(x_2)$ is achieved at $x_2 = \frac{1}{\sqrt{2}}$, which yields $f_2(\frac{1}{\sqrt{2}}) = 2^{-\frac{p}{2}+1}$.

(c) Assuming that (1) is true for $n = i - 1$, we now show that it is true for $n = i$. For $n = i$, (1) can be written as

$$\begin{aligned} \min \quad & x_i^p + \sum_{j=1}^{i-1} x_j^p \\ \text{s.t.} \quad & x_i^2 + \sum_{j=1}^{i-1} x_j^2 = 1. \quad x_i \ge 0, \quad x_j \ge 0 \text{ for } j = 1, \dots, i-1. \end{aligned}$$

We need two steps to solve the problem. In the first step, we fix $0 \le x_i \le 1$, and find the best $x_1, \dots, x_{i-1}$. In the second step, we will find the best x_i. Suppose now x_i has been fixed, the problem we are considering is

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{i-1} x_i^p \\
\text{s.t.} \quad & \sum_{j=1}^{i-1} x_j^2 = 1 - x_i^2, \\
& x_j \geq 0 \text{ for } j = 1, \ldots, i-1.
\end{aligned}
$$

If $x_i = 1$, the result of the optimization is zero. If $0 \leq x_i < 1$, after the change of variable $\tilde{x}_j = \frac{1}{\sqrt{1-x_i^2}} x_j$, the problem is turned to

$$
\begin{aligned}
\min \quad & (1 - x_i^2)^{\frac{p}{2}} \sum_{j=1}^{i-1} \tilde{x}_j^p \\
\text{s.t.} \quad & \sum_{j=1}^{i-1} \tilde{x}_j^2 = 1, \\
& \tilde{x}_j \geq 0 \text{ for } j = 1, \ldots, i-1,
\end{aligned}
$$

which is just $(1 - x_i^2)^{\frac{p}{2}} (i-1)^{-\frac{p}{2}+1}$, by using the assumption that (1) is just $(i-1)^{-\frac{p}{2}+1}$ for $n = i - 1$.
Now, as the second step, we need to solve

$$
\begin{aligned}
\min \quad & f(x_i) \triangleq x_i^p + (1 - x_i^2)^{\frac{p}{2}} (i-1)^{-\frac{p}{2}+1} \\
\text{s.t.} \quad & 0 \leq x_i \leq 1.
\end{aligned}
$$

This again can be solved by observing the derivative, which can be shown to be

$$
f_i'(x_i) = p x_i (x_i^{p-2} - (i-1)^{1-\frac{p}{2}} (1 - x_i^2)^{\frac{p}{2}-1}).
$$

It can be seen that

$$
f_2'(x_1) \begin{cases} < 0, & 0 \leq x_i < \frac{1}{\sqrt{i}}, \\ = 0, & x_i = \frac{1}{\sqrt{i}}, \\ > 0, & \frac{1}{\sqrt{2}} < x_i \leq 1. \end{cases}
$$

Therefore, $x_i = \frac{1}{\sqrt{i}}$ yields the minimum value $f_i(\frac{1}{\sqrt{i}}) = i^{-\frac{p}{2}+1}$.