Tutorial 10: Application of convex optimization in FIR filter design

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FIR filter

- An FIR filter can be represented by its impulse response:

$$h[n] = h_0 + h_1\delta[n-1] + \ldots + h_{L-1}\delta[n-L+1],$$

  - L is the filter length
  - $\boldsymbol{h} = [h_0, \ldots, h_{L-1}]^T \in \mathbb{R}^L$ is the filter coefficient
  - $\delta[n]$ is the delta function

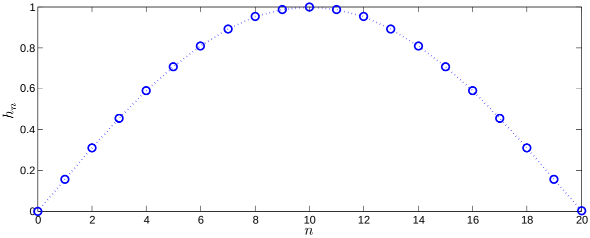
- Equivalently, an FIR filter can be represented by its frequency response:

$$H(\omega) = h_0 + h_1 e^{-j\omega} + \ldots + h_{L-1} e^{-j(L-1)\omega}$$

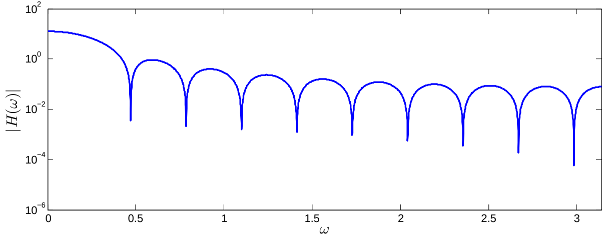
  - ◇ $j = \sqrt{-1}$.
  - ◇ As $H(\omega + 2\pi) = H(\omega)$ and $H(-\omega) = H^\star(\omega)$, we only need to specify $H(\omega)$ for $0 \leq \omega \leq \pi$.

Example

- Impulse repsonse $h[n]$



- Magnitude $|H(\omega)|$



Design objective

- The objective is to choose $\boldsymbol{h}$ such that $H(\omega)$ satisfies some specifications.
- Chebychev filter design:

$$|H(\omega) - D(\omega)| \leq E(\omega) \text{ for all } \omega \in [0, \pi],$$

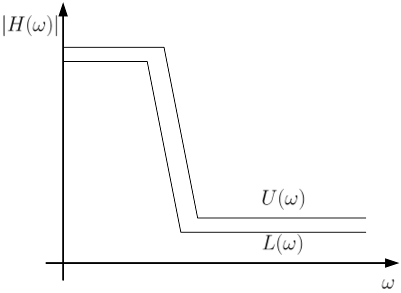
where $D(\omega)$ is the desired freq. response, and $E(\omega)$ is error tolerance.

- Magnitude filter design:

$$L(\omega) \leq |H(\omega)| \leq U(\omega) \text{ for all } \omega \in [0, \pi].$$

where $L(\omega)$ and $U(\omega)$ are given lower and upper freq. response bounds.

- Magnitude filter design



- Recall that $H(\omega)$ is a linear function of $\boldsymbol{h}$.
- Chebychev filter design:

$$|H(\omega) - D(\omega)| \leq E(\omega) \text{ for all } \omega \in [0, \pi],$$

is convex (in $\boldsymbol{h}$) for each ω.

- Magnitude filter design:

$$L(\omega) \leq |H(\omega)| \leq U(\omega) \text{ for all } \omega \in [0, \pi].$$

is not convex generally.

- Both are semi-infinite problems, i.e. they involve infinite number of constraints.

Handinng nonconvexity in magnitude filter design

- Magnitude filter design:

$$\begin{aligned} \min_{\boldsymbol{h}} \quad & f_0(\boldsymbol{h}) \\ \text{s.t.} \quad & L(\omega) \leq |H(\omega)| \leq U(\omega) \text{ for all } \omega \in [0, \pi]. \end{aligned}$$

  where $f_0(\boldsymbol{h})$ is convex.

- How to handle the nonconvex constraint $L(\omega) \leq |H(\omega)|$?
  - ◇ Method 1: Impose the linear phase structure on $\boldsymbol{h}$, i.e.

$$h_n = h_{L-1-n}, \quad n = 0, \ldots, L$$

    where the filter order $L = 2l + 1$ is assumed to be odd.

  - ◇ Method 2: Optimize the autocorrelation $\boldsymbol{r}$ instead of $\boldsymbol{h}$, where

$$r_\tau = \sum_{n=-L+1}^{L-1} h_n h_{n+\tau}, \quad \tau = 0, \ldots, L-1.$$

Linear phase filters

- Linear phase structure

$$h_n = h_{L-1-n}, \quad n = 0, \ldots, L$$

where the filter order $L = 2l + 1$ is assumed to be odd.

- The freq. response is rewritten as

$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-j\omega} + \ldots + h_{L-1} e^{-j(L-1)\omega} \\ &= e^{-jl\omega}(2h_0 \cos l\omega + 2h_1 \cos(l-1)\omega + \ldots + h_l) \\ &\triangleq e^{-jl\omega} \tilde{H}(\omega) \end{aligned}$$

- Observations:
  - ◇ The phase is linear in ω, except for jumps of $\pm\pi$.
  - ◇ $\tilde{H}(\omega)$ is real and only depends on $\tilde{\boldsymbol{h}} \in \mathbf{R}^{l+1}$ which collects the first half of entries of $\boldsymbol{h}$.
  - ◇ $|H(\omega)| = |\tilde{H}(\omega)|$.

Linear phase filters, cont'd

- Linear phase filter design:

$$\begin{aligned} \min_{\tilde{\boldsymbol{h}}} \quad & f_0(\tilde{\boldsymbol{h}}) \\ \text{s.t.} \quad & L(\omega) \leq |\tilde{H}(\omega)| \leq U(\omega), \quad \forall\, \omega \in [0, \pi] \end{aligned}$$

- As $\tilde{H}(\omega)$ is real and linear in $\tilde{\boldsymbol{h}}$, the constraint is convex (with infinite number of constraints).

Autocorrelation-based filter design

- The autocorrelation $\boldsymbol{r}$ is defined as

$$r_\tau = \sum_{n=-L+1}^{L-1} h_n h_{n+\tau}, \quad \tau = 0, \ldots, L-1.$$

- Fourier transform of $\boldsymbol{r}$ is

$$R(\omega) = \sum_\tau e^{-j\omega\tau} r_\tau = r_0 + \sum_{\tau=1}^{L-1} 2 r_\tau \cos \omega\tau = |H(\omega)|^2.$$

- As $R(\omega)$ is linear in $\boldsymbol{r}$,

$$L^2(\omega) \leq R(\omega) \leq U^2(\omega)$$

is convex.

Autocorrelation-based filter design, cont'd

- But not all given $\boldsymbol{r}$ is valid autocorrelation of some $\boldsymbol{h}$.
- Spectral factorization theorem: $\boldsymbol{r}$ is a valid autocorrelation if and only if

$$R(\omega) \geq 0, \quad \omega \in [0, \pi].$$

Autocorrelation-based filter design, cont'd

- Autocorrelation-based filter design:

$$
\begin{aligned}
\min_{\boldsymbol{r}} \quad & \tilde{f}_0(\boldsymbol{r}) \\
\text{s.t.} \quad & L^2(\omega) \leq R(\omega) \leq U^2(\omega), && \forall\, \omega \in [0, \pi] \\
& R(\omega) \geq 0, && \forall\, \omega \in [0, \pi].
\end{aligned}
$$

- Use spectral factorization to recover $\boldsymbol{h}$ from the solution $\boldsymbol{r}^\star$.

Convex semi-infinite programming

- Consider the general semi-infinite problem

$$\begin{aligned} \min_{x} \quad & f_0(x) \\ \text{s.t.} \quad & Ax = b, \\ & f_i(x) \le 0, \qquad i = 1, \dots, m, \\ & g_i(x, \omega) \le 0, \quad \omega \in [0, \pi],\ i = 1, \dots, p \end{aligned}$$

  where we assume $f_i(x)$ and $g_i(x, \omega)$ are convex for every ω.

- x may have the meaning of filter coefficients $\boldsymbol{h}$, or autocorrelaction $\boldsymbol{r}$, or other variables.

Handling semi-infinte constraints

- How to handle infinite number of constraint?
- Method 1: $g_i(x, \omega) \leq 0, \ \forall \ \omega \in [0, \pi]$ is the same as

$$h_i(x) \triangleq \sup_{\omega \in [0,\pi]} g_i(x, \omega) \leq 0.$$

- $h_i(x)$ is convex.
  - ◇ If $h_i(x)$ admits closed-form solution, then we are lucky.
  - ◇ If $h_i(x)$ and a $\bar{\omega}$ such that $h_i(x) = g_i(x, \bar{\omega})$ can be computed efficiently, methods for nondifferential optimization can be used.
  - ◇ If $h_i(x)$ admits good approximation, such as one-dimension search on ω, use the approximation.
  - ◇ If all fail, consider modifying your specification.

- Method 2: Sampling on frequency, i.e. choose some

$$0 \leq \omega_1 < \omega_2, < \ldots < \omega_K \leq \pi,$$

and impose the constraints

$$g_i(x, \omega_k) \leq -\epsilon, \quad k = 1, \ldots, K.$$

- $\{\omega_k\}_{k=1}^K$ can be chosen uniformly or logarithmically spaced.
- $\epsilon \geq 0$ is chosen to ensure

$$g_i(x, \omega) \leq 0, \ \forall \ \omega \in [0, \pi].$$

- Engineering rule of thumb: $K = 15L$ when using uniform spacing.
- This method is an approximation, but flexible, and often yields good results.

- Method 3: The Kalman-Yakubovich-Popov (KYP) lemma can transform some particular form of semi-infinite constraints to some finite number of constraints involving linear matrix inequalities.

- KYP lemma: Given A, B, and a Hermitian matrix Θ,

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^H \Theta \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0$$

  holds for all $\omega \in \mathbb{R} \cup \{\infty\}$ if and only if there exists a Hermitian matrix P s/t

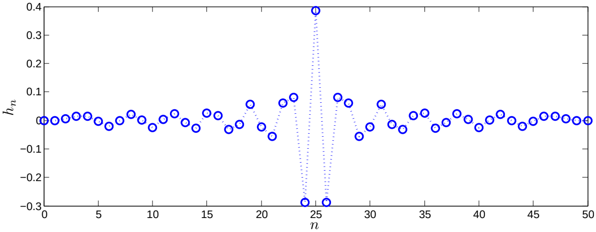
$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^H \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}^H \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta \prec 0.$$

- There are generalized KYP lemmas which can be very useful.

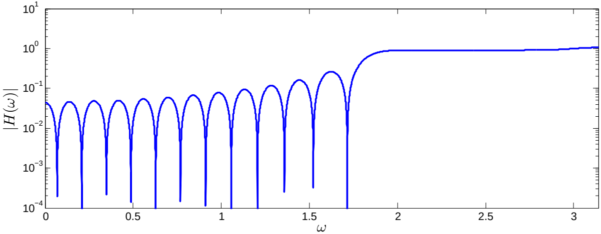
- KYP lemma is an exact transformation, but requires some specific form of semi-infinite constraints.

Example: Linear phase high pass filter

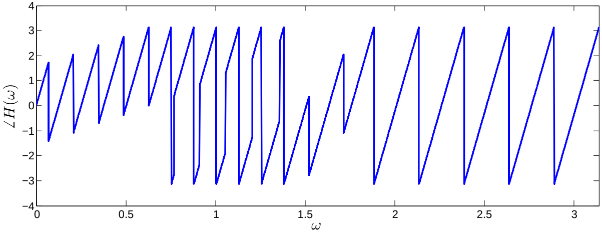
- Specification:
  - ◇ Linear phase filter, $L = 51$.
  - ◇ Passband $[0.8\pi, \pi]$; Stopband $[0, 0.7\pi]$.
  - ◇ Constraint: Passband magnitude $1/1.01 \leq |\tilde{H}(\omega)| \leq 1.01$.
  - ◇ Objective: Minimize stopband magnitude.
- Time response $h[n]$



- Magnitude $|H(\omega)|$



- Phase $\angle H(\omega)$



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