Tutorial 10: Application of convex optimization in FIR filter design

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An FIR filter can be represented by its impulse response:

\[ h[n] = h_0 + h_1 \delta[n - 1] + \ldots + h_{L-1} \delta[n - L + 1], \]

- \( L \) is the filter length
- \( h = [h_0, \ldots, h_{L-1}]^T \in \mathbb{R}^L \) is the filter coefficient
- \( \delta[n] \) is the delta function
Equivalently, an FIR filter can be represented by its frequency response:

\[ H(\omega) = h_0 + h_1 e^{-j\omega} + \ldots + h_{L-1} e^{-j(L-1)\omega} \]

\[ j = \sqrt{-1}. \]

As \( H(\omega + 2\pi) = H(\omega) \) and \( H(-\omega) = H^*(\omega) \), we only need to specify \( H(\omega) \) for \( 0 \leq \omega \leq \pi \).
Example

- Impulse response $h[n]$

- Magnitude $|H(\omega)|$
Design objective

- The objective is to choose \( h \) such that \( H(\omega) \) satisfies some specifications.

- Chebychev filter design:

\[
|H(\omega) - D(\omega)| \leq E(\omega) \text{ for all } \omega \in [0, \pi],
\]

where \( D(\omega) \) is the desired freq. response, and \( E(\omega) \) is error tolerance.

- Magnitude filter design:

\[
L(\omega) \leq |H(\omega)| \leq U(\omega) \text{ for all } \omega \in [0, \pi].
\]

where \( L(\omega) \) and \( U(\omega) \) are given lower and upper freq. response bounds.
Magnitude filter design
• Recall that $H(\omega)$ is a linear function of $h$.

• Chebychev filter design:

\[ |H(\omega) - D(\omega)| \leq E(\omega) \text{ for all } \omega \in [0, \pi], \]

is convex (in $h$) for each $\omega$.

• Magnitude filter design:

\[ L(\omega) \leq |H(\omega)| \leq U(\omega) \text{ for all } \omega \in [0, \pi]. \]

is not convex generally.

• Both are semi-infinite problems, i.e. they involve infinite number of constraints.
Handling nonconvexity in magnitude filter design

- Magnitude filter design:

\[
\begin{align*}
\min_{\mathbf{h}} & \quad f_0(\mathbf{h}) \\
\text{s.t.} & \quad L(\omega) \leq |H(\omega)| \leq U(\omega) \text{ for all } \omega \in [0, \pi].
\end{align*}
\]

where \( f_0(\mathbf{h}) \) is convex.

- How to handle the nonconvex constraint \( L(\omega) \leq |H(\omega)| \)?

  ◦ Method 1: Impose the linear phase structure on \( \mathbf{h} \), i.e.

  \[
  h_n = h_{L-1-n}, \quad n = 0, \ldots, L
  \]

  where the filter order \( L = 2l + 1 \) is assumed to be odd.

  ◦ Method 2: Optimize the autocorrelation \( \mathbf{r} \) instead of \( \mathbf{h} \), where

  \[
  r_\tau = \sum_{n=-L+1}^{L-1} h_n h_{n+\tau}, \quad \tau = 0, \ldots, L - 1.
  \]
Linear phase filters

- Linear phase structure

\[ h_n = h_{L-1-n}, \quad n = 0, \ldots, L \]

where the filter order \( L = 2l + 1 \) is assumed to be odd.

- The freq. response is rewritten as

\[
H(\omega) = h_0 + h_1 e^{-j\omega} + \ldots + h_{L-1} e^{-j(L-1)\omega} \\
= e^{-jl\omega} (2h_0 \cos l\omega + 2h_1 \cos(l - 1)\omega + \ldots + h_l) \\
\triangleq e^{-jl\omega} \tilde{H}(\omega)
\]

- Observations:
  - The phase is linear in \( \omega \), except for jumps of \( \pm \pi \).
  - \( \tilde{H}(\omega) \) is real and only depends on \( \tilde{h} \in \mathbb{R}^{l+1} \) which collects the first half of entries of \( h \).
  - \( |H(\omega)| = |\tilde{H}(\omega)| \).
Linear phase filters, cont’d

- Linear phase filter design:

\[
\begin{align*}
&\min_{\tilde{h}} f_0(\tilde{h}) \\
&\text{s.t. } L(\omega) \leq |\tilde{H}(\omega)| \leq U(\omega), \quad \forall \omega \in [0, \pi]
\end{align*}
\]

- As $\tilde{H}(\omega)$ is real and linear in $\tilde{h}$, the constraint is convex (with infinite number of constraints).
The autocorrelation $r$ is defined as

$$r_{\tau} = \sum_{n=-L+1}^{L-1} h_n h_{n+\tau}, \quad \tau = 0, \ldots, L - 1.$$ 

Fourier transform of $r$ is

$$R(\omega) = \sum_{\tau} e^{-j\omega \tau} r_{\tau} = r_0 + \sum_{\tau=1}^{L-1} 2r_{\tau} \cos \omega \tau = |H(\omega)|^2.$$ 

As $R(\omega)$ is linear in $r$,

$$L^2(\omega) \leq R(\omega) \leq U^2(\omega)$$

is convex.
But not all given $r$ is valid autocorrelation of some $h$.

Spectral factorization theorem: $r$ is a valid autocorrelation if and only if

$$R(\omega) \geq 0, \quad \omega \in [0, \pi].$$
Autocorrelation-based filter design, cont’d

Autocorrelation-based filter design:

\[
\begin{align*}
\min_{r} & \quad \tilde{f}_0(r) \\
\text{s.t.} & \quad L^2(\omega) \leq R(\omega) \leq U^2(\omega), \quad \forall \omega \in [0, \pi] \\
& \quad R(\omega) \geq 0, \quad \forall \omega \in [0, \pi].
\end{align*}
\]

Use spectral factorization to recover \( h \) from the solution \( r^* \).
Consider the general semi-infinite problem

\[
\begin{align*}
\min_{x} & \quad f_0(x) \\
\text{s.t.} & \quad Ax = b, \\
& \quad f_i(x) \leq 0, \quad i = 1, \ldots, m, \\
& \quad g_i(x, \omega) \leq 0, \quad \omega \in [0, \pi], \ i = 1, \ldots, p
\end{align*}
\]

where we assume $f_i(x)$ and $g_i(x, \omega)$ are convex for every $\omega$.

$x$ may have the meaning of filter coefficients $h$, or autocorrelation $r$, or other variables.
Handling semi-infinite constraints

- How to handle infinite number of constraints?

- Method 1: \( g_i(x, \omega) \leq 0, \ \forall \ \omega \in [0, \pi] \) is the same as

\[
h_i(x) \triangleq \sup_{\omega \in [0, \pi]} g_i(x, \omega) \leq 0.
\]

- \( h_i(x) \) is convex.
  - If \( h_i(x) \) admits closed-form solution, then we are lucky.
  - If \( h_i(x) \) and a \( \bar{\omega} \) such that \( h_i(x) = g_i(x, \bar{\omega}) \) can be computed efficiently, methods for nondifferential optimization can be used.
  - If \( h_i(x) \) admits good approximation, such as one-dimension search on \( \omega \), use the approximation.
  - If all fail, consider modifying your specification.
• Method 2: Sampling on frequency, i.e. choose some

\[ 0 \leq \omega_1 < \omega_2, < \ldots < \omega_K \leq \pi, \]

and impose the constraints

\[ g_i(x, \omega_k) \leq -\epsilon, \quad k = 1, \ldots, K. \]

\{\omega_k\}_{k=1}^{K} can be chosen uniformly or logarithmically spaced.

\[ \epsilon \geq 0 \] is chosen to ensure

\[ g_i(x, \omega) \leq 0, \quad \forall \omega \in [0, \pi]. \]

• Engineering rule of thumb: \( K = 15L \) when using uniform spacing.

• This method is an approximation, but flexible, and often yields good results.
Method 3: The Kalman-Yakubovich-Popov (KYP) lemma can transform some particular form of semi-infinite constraints to some finite number of constraints involving linear matrix inequalities.

KYP lemma: Given $A, B$, and a Hermitian matrix $\Theta$,

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^H \Theta \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0$$

holds for all $\omega \in \mathbb{R} \cup \{\infty\}$ if and only if there exists a Hermitian matrix $P$ s/t

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^H \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}^H \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta \prec 0.$$

There are generalized KYP lemmas which can be very useful.

KYP lemma is an exact transformation, but requires some specific form of semi-infinite constraints.
Example: Linear phase high pass filter

- **Specification:**
  - Linear phase filter, \( L = 51 \).
  - Passband \([0.8\pi, \pi]\); Stopband \([0, 0.7\pi]\).
  - Constraint: Passband magnitude \( 1/1.01 \leq |\tilde{H}(\omega)| \leq 1.01 \).
  - Objective: Minimize stopband magnitude.

- **Time response** \( h[n] \)
- **Magnitude** $|H(\omega)|$

![Magnitude Graph](image1)

- **Phase** $\angle H(\omega)$

![Phase Graph](image2)

S. Boyd, “Filter design”, Lecture slide of EE364a, Stanford University.

