ELEG5481

SIGNAL PROCESSING OPTIMIZATION TECHNIQUES

6. GEOMETRIC PROGRAM

Some Basics

- A function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ with $\mathbf{dom} f = \mathbf{R}^n_{++}$, defined as

$$f(x) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

  where $c > 0$ & $a_i \in \mathbf{R}$, is called a **monomial function**.

- A sum of monomials

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$$

  where $c_k > 0$ & $a_{ik} \in \mathbf{R}$, is called a **posynomial function**.

- The **log-sum-exp function**

$$f(x) = \log(e^{x_1} + \cdots + e^{x_n}),$$

  where $\mathbf{dom} f = \mathbf{R}^n$.

- The log-sum-exp function is convex on $\mathbf{R}^n$.

- The following extended log-sum-exp function

$$f(x) = \log(e^{a_1^T x + b_1} + \cdots + e^{a_m^T x + b_m}),$$

  where $a_i \in \mathbf{R}^n$ & $b_i \in \mathbf{R}$, is also convex on $\mathbf{R}^n$.

Geometric Program

$$
\begin{array}{rl}
\min & \sum_{k=1}^{K_0} c_{0,k} x_1^{a_{0,1k}} x_2^{a_{0,2k}} \cdots x_n^{a_{0,nk}} \\
\text{s.t.} & \sum_{k=1}^{K_i} c_{i,k} x_1^{a_{i,1k}} x_2^{a_{i,2k}} \cdots x_n^{a_{i,nk}} \leq 1, \quad i = 1, \ldots, m \\
& d_i x_1^{g_{i1}} \cdots x_n^{g_{in}} = 1, \quad i = 1, \ldots, p
\end{array}
$$

where $\mathcal{D} = \mathbf{R}_{++}^n$.

- The objective & inequality constraint functions are posynomials.
- The equality constraint functions are monomials.
- Not a convex optimization problem in general.

Geometric Program in a Convex Form

- Let $x_i = e^{y_i}$.
- A monomial can be re-expressed as

$$cx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} = ce^{a_1 y_1} \cdots e^{a_n y_n} = e^{a^T y + b}$$

  where $b = \log c$.

- Likewise, a posynomial can be re-expressed as

$$\sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}} = \sum_{k=1}^{K} e^{a_k^T y + b_k}$$

  where $b_k = \log c_k$.

- The geometric program can then be transformed to

$$\begin{aligned} \min \ & \textstyle\sum_{k=1}^{K_0} e^{a_{0k}^T y + b_{0k}} \\ \text{s.t.} \ & \textstyle\sum_{k=1}^{K_i} e^{a_{ik}^T y + b_{ik}} \leq 1, \quad i = 1, \ldots, m \\ & e^{g_i^T y + h_i} = 1, \quad i = 1, \ldots, p \end{aligned}$$

- By applying the logarithm to all functions, we obtain an equiv. problem

$$
\begin{aligned}
\min \;& \log \textstyle\sum_{k=1}^{K_0} e^{a_{0k}^T y + b_{0k}} \\
\text{s.t.} \;& \log \textstyle\sum_{k=1}^{K_i} e^{a_{ik}^T y + b_{ik}} \leq 0, \quad i = 1, \ldots, m \\
& g_i^T y + h_i = 0, \quad i = 1, \ldots, p
\end{aligned}
$$

  that is convex.

Example: Optimal Power Assignment Revisited

- Recall the power assignment problem in the last lesson:

$$\begin{aligned} \min \quad & \max_{i=1,\ldots,K} \frac{\sum_{j \neq i} G_{ij} p_j + \sigma_i^2}{G_{ii} p_i} \\ \text{s.t.} \; & 0 \leq p_i \leq p_{\max,i}, \quad i = 1, \ldots, K \end{aligned}$$

- By epigraph reformulation, we may rewrite the problem as

$$\begin{aligned} \min \; & t \\ \text{s.t.} \; & \frac{\sum_{j \neq i} G_{ij} p_j + \sigma_i^2}{G_{ii} p_i} \leq t, \quad i = 1, \ldots, K \\ & 0 \leq p_i \leq p_{\max,i}, \quad i = 1, \ldots, K \end{aligned}$$

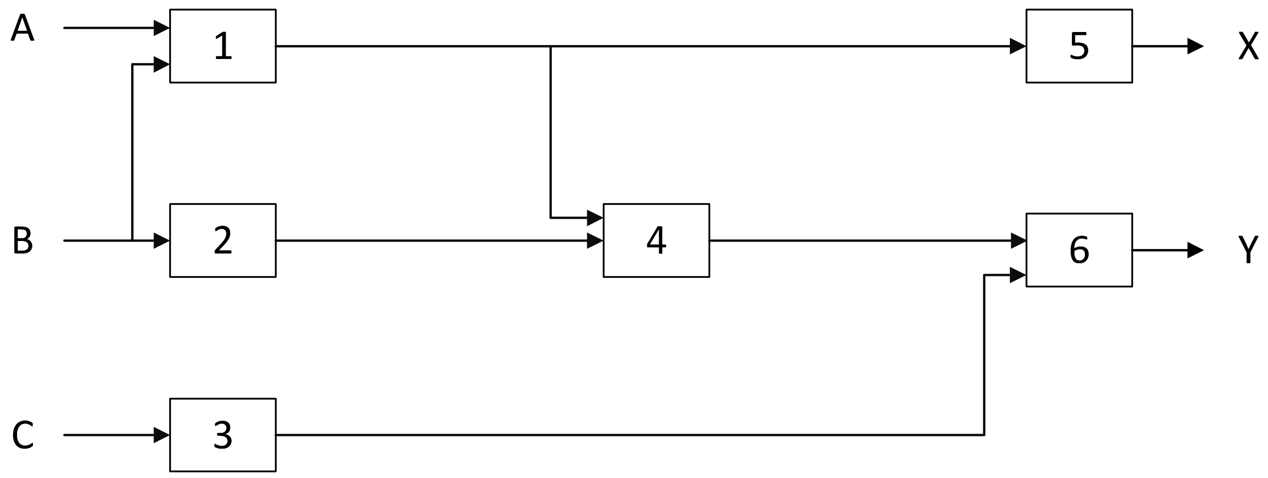
- The problem can be further reformulated as

$$
\begin{aligned}
&\min\ t \\
&\text{s.t. } \sum_{j \neq i} G_{ij} G_{ii}^{-1} p_j p_i^{-1} t^{-1} + \sigma_i^2 G_{ii}^{-1} p_i^{-1} t^{-1} \leq 1,\ i = 1, \ldots, K \\
&\qquad p_{\max,i}^{-1} p_i \leq 1, \qquad i = 1, \ldots, K
\end{aligned}
$$

with $\mathcal{D} = \mathbf{R}_{++}^{K+1}$. The problem becomes a geometric program.

Example: Gate sizing

- Consider a digital circuit with gates interconnected by wires.
- The problem is to determine the size of each gate to minimize the delay in the circuit subject to area and power constraints.



- The total area of all gates is

$$A = \sum_{i=1}^{n} a_i x_i,$$

where $x_i \geq 1$ is the scale factor of gate i, and a_i is the area of gate i under unit scaling.

- The total power consumed by all gates is

$$P = \sum_{i=1}^{n} f_i e_i x_i,$$

where f_i is the transition frequency of the gate, and e_i is the energy lost in gate transition.

- The input capacitance C_i and the driving resistance R_i can be modeled as

$$C_i = \alpha_i + \beta_i x_i, \qquad R_i = \gamma_i / x_i,$$

for some positive constants α_i, β_i and γ_i.

- The delay of a gate is

$$D_i = \begin{cases} R_i \sum_{j \in F(i)} C_j, & \text{if } i \text{ is not an output gate} \\ R_i C_i^{\text{out}}, & \text{if } i \text{ is an output gate} \end{cases}$$

  where $F_{(i)}$ is the set of gates with the inputs connected to gate i, and C_i^{out} is the load capacitance of an output gate.

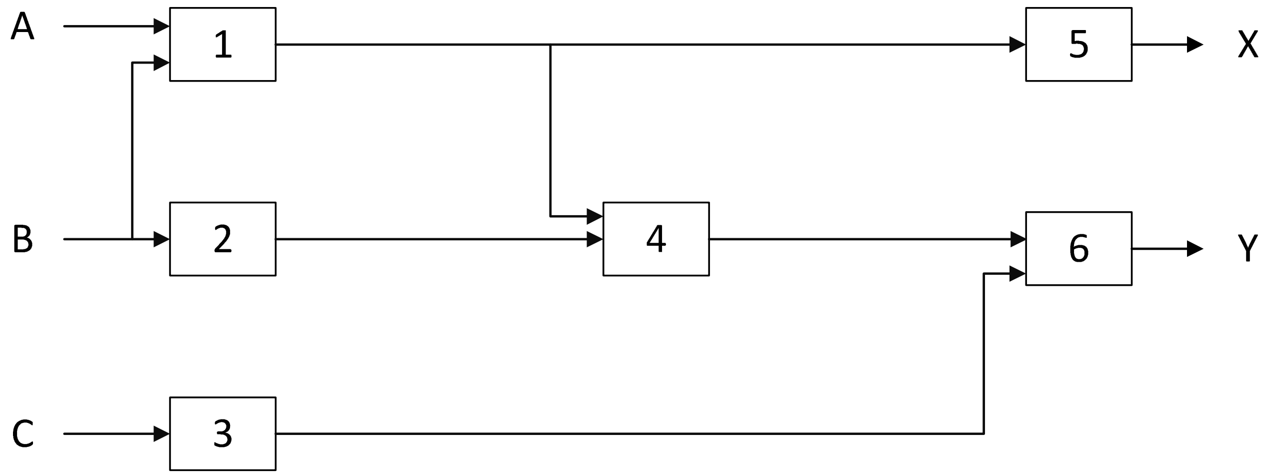
- The worst-case delay D is the maximum delay of all paths from inputs to outputs, i.e.

$$D = \max_{I \in \mathcal{O}} \sum_{i \in I} D_i,$$

  where $\mathcal{O}$ is the set of all paths from inputs to outputs.

- In this example, D is given by

$$D = \max\{D_1 + D_5, D_1 + D_4 + D_6, D_2 + D_4 + D_6, D_3 + D_6\}.$$



- The problem can be formulated as

$$\begin{aligned} \min\ & D \\ \text{s.t. } & A \leq A_{\max}, \quad P \leq P_{\max}, \\ & x_i \geq 1,\ i = 1, \ldots, n. \end{aligned}$$

- This problem is not a GP, as D is the maximum of some posynomials.
- But it can be converted to a GP by adding a slack variable t,

$$\begin{aligned} \min\ & t \\ \text{s.t. } & \sum_{i \in I} D_i \leq t,\ \text{for all } I \in \mathcal{O}, \\ & A \leq A_{\max}, \quad P \leq P_{\max}, \\ & x_i \geq 1,\ i = 1, \ldots, n. \end{aligned}$$

Additional Reading

S. Boyd, S.-J. Kim, L. Vandenberghe, and A. Hassibi, "A tutorial on geometric programming," *Optimization and Engineering*, 2007

M. Chiang, "Geometric programming for communication systems", *Foundations and Trends in Communications and Information Theory*, 2005.

M. Chiang, C. W. Tan, D. P. Palomar, D. O'Neill, and D. Julian, "Power control by geometric programming," *IEEE Trans. Wireless Commun.*, 2007.