ELE5481

SIGNAL PROCESSING OPTIMIZATION TECHNIQUES

5. LINEAR AND QUADRATIC PROGRAMS

Linear Programming (LP)

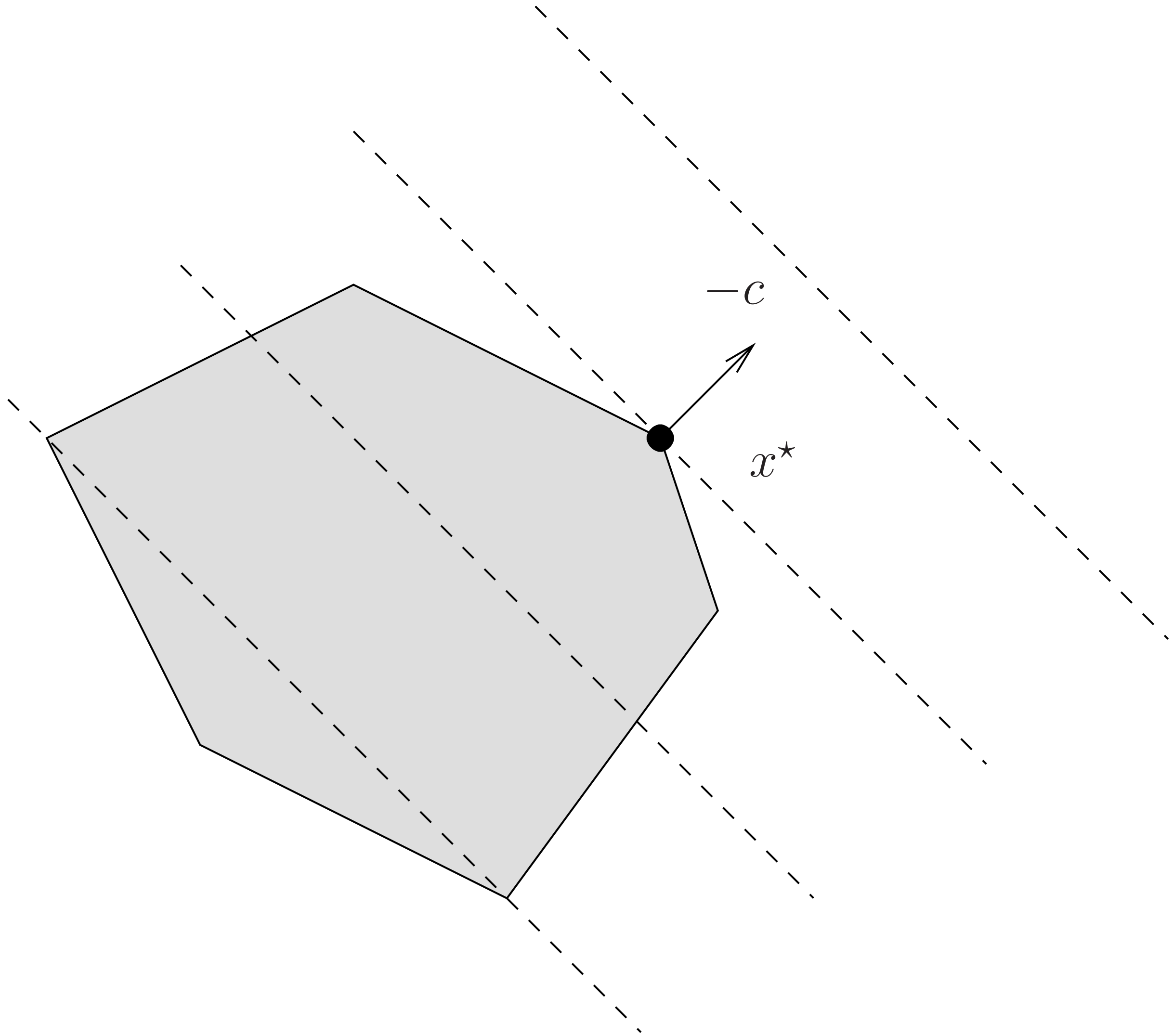
A general form of LP:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Gx \preceq h \\ & Ax = b \end{aligned} \tag{1}$$

A standard form of LP widely used in the literature & software:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & x \succeq 0 \\ & Ax = b \end{aligned} \tag{2}$$

LP is a problem of minimizing a linear objective function over a polyhedron.



The general form in (1) can be reformulated as the standard form in (2). Problem (1) is equiv. to

$$\begin{aligned} &\min\ c^T x \\ &\text{s.t. } s \succeq 0, \quad h - Gx = s, \quad Ax = b \end{aligned} \tag{$*$}$$

Let $x = x^+ - x^-$, where $x^+, x^- \succeq 0$. Eq. $(*)$ is equiv. to

$$\min\ [\ c^T \ \ -c^T \ \ 0\ ] \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix}$$

$$\text{s.t. } \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} \succeq 0, \quad \begin{bmatrix} A & -A & 0 \\ G & -G & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = \begin{bmatrix} b \\ h \end{bmatrix}$$

A Brief History

- 1939: planning, production (Kantorovich)
  - Noble prize in Economics, 1975
- 1940's: simplex algorithm for LP (Dantzig)
  - used in Berlin airlift, 1948
- 1970's: polynomial-time ellipsoid algorithm for LP (Khachiyan)
  - based on work by Dirkin, Shor and Nemirovski in 1960's
  - front page news in the Western world, incl. New York Times (exaggerated)
- 1980's: polynomial-time interior-point algorithm for LP (Karmarkar)
- late 1980's-now: polynomial-time interior-point methods for nonlinear convex programs (Nesterov and Nemirovski, 1994)
  - convex opt. software we used today is largely based on interior-point methods.

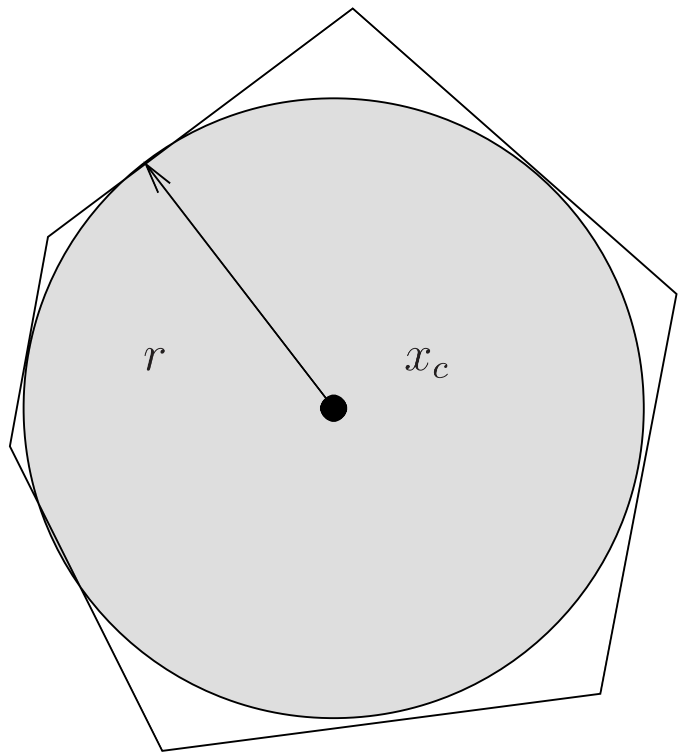
Application: Diet Problem

- x_i is the quantity of food i.
- Each unit of food i has a cost of c_i.
- One unit of food j contains an amount A_{ij} of nutrient j.
- We want nutrient i to be at least equal to b_i.
- **Problem:** find the cheapest diet such that the minimum nutrient requirements are fulfilled.
- This problem can be cast as an LP:

$$\begin{aligned} &\min\ c^T x \\ &\text{ s.t. } Ax \preceq b, \quad x \succeq 0 \end{aligned}$$

Chebychev Center

- Let a norm ball $B(x_c, r) = \{ \, x \mid \|x_c - x\|_2 \leq r \, \}$, & a polyhedron $\mathcal{P} = \{ \, x \mid a_i^T x \leq b_i, \;\; i = 1, \ldots, m \, \}$.
- **Problem:** Find the largest ball inside a polyhedron $\mathcal{P}$; i.e., $\max_{x_c, r} \; r$, subject to $B(x_c, r) \subseteq \mathcal{P}$.



- An alternative representation of the norm ball: $B(x_c, r) = \{ \, x_c + u \mid \|u\|_2 \leq r \, \}$.

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$$B(x_c, r) \subseteq \mathcal{P} \Longleftrightarrow \sup_u \{ \, a_i^T (x_c + u) \mid \|u\|_2 \leq r \, \} \leq b_i, \quad \forall i$$
$$\Longleftrightarrow a_i^T x_c + r\|a_i\|_2 \leq b_i, \quad \forall i$$

- Hence, the Chebychev center problem is equiv. to an LP

$$\begin{aligned} &\max \; r \\ &\;\text{s.t.} \; a_i^T x_c + r\|a_i\|_2 \leq b_i, \quad i = 1, \ldots, m \end{aligned}$$

Piecewise Linear Minimization

$$\min \max_{i=1,\ldots,m} (a_i^T x + b)$$

By using the epigraph form, the problem is equiv. to

$$\begin{aligned} & \min t \\ & \quad \text{s.t. } \max_{i=1,\ldots,m}(a_i^T x + b) \leq t \\ \Longleftrightarrow & \min t \\ & \quad \text{s.t. } a_i^T x + b \leq t, \quad i = 1, \ldots, m \end{aligned}$$

which is an LP.

ℓ_∞-norm (Chebychev) Approximation

$$\min \|Ax - b\|_\infty$$

Using the epigraph form, the ℓ_∞-norm approx. problem can be cast as an LP:

$$\begin{aligned} & \min\ t \\ & \quad \text{s.t.}\ \ \max_{i=1,\ldots,m} |r_i| \le t \\ & \qquad\quad r = Ax - b \\ \iff & \min\ t \\ & \quad \text{s.t.}\ \ -t\mathbf{1} \preceq r \preceq t\mathbf{1} \\ & \qquad\quad r = Ax - b \end{aligned}$$

ℓ_1-norm Approximation

$$\min \ \|Ax - b\|_1$$

can be rewritten as

$$\begin{aligned} &\min \ \textstyle\sum_{i=1}^m |r_i| \\ &\ \text{s.t.} \ r = Ax - b \\ \iff &\min \ \textstyle\sum_{i=1}^m t_i \\ &\ \text{s.t.} \ -t_i \leq r_i \leq t_i, \quad i = 1, \ldots, m \\ &\qquad\ r = Ax - b \end{aligned}$$

which is an LP.

Linear Fractional Programming

$$
\begin{aligned}
\min \ & \frac{c^T x + d}{f^T x + g} \\
\text{s.t.} \ & Ax \preceq b \\
& f^T x + g > 0
\end{aligned}
$$

- The objective function is quasiconvex, and each of its sublevel sets is a polyhedron.

If the feasible set $\{x | Ax \preceq b, f^T x + g > 0\}$ is bounded, the linear fractional program can be transformed to an equiv. LP:

$$\begin{aligned} \min_{y \in \mathbf{R}^n, z \in \mathbf{R}} \quad & c^T y + dz \\ \text{s.t.} \quad & Ay - bz \preceq 0, \quad z \geq 0 \\ & f^T y + gz = 1 \end{aligned}$$

- It can be shown that if $\{x | Ax \preceq b, f^T x + g > 0\}$ is bounded, then $z > 0$ for any feasible (y, z).
- If (y, z) is feasible in the LP, then $x = y/z$ is feasible in the linear fractional program.
- This transformation is known as the Charnes-Cooper transformation.

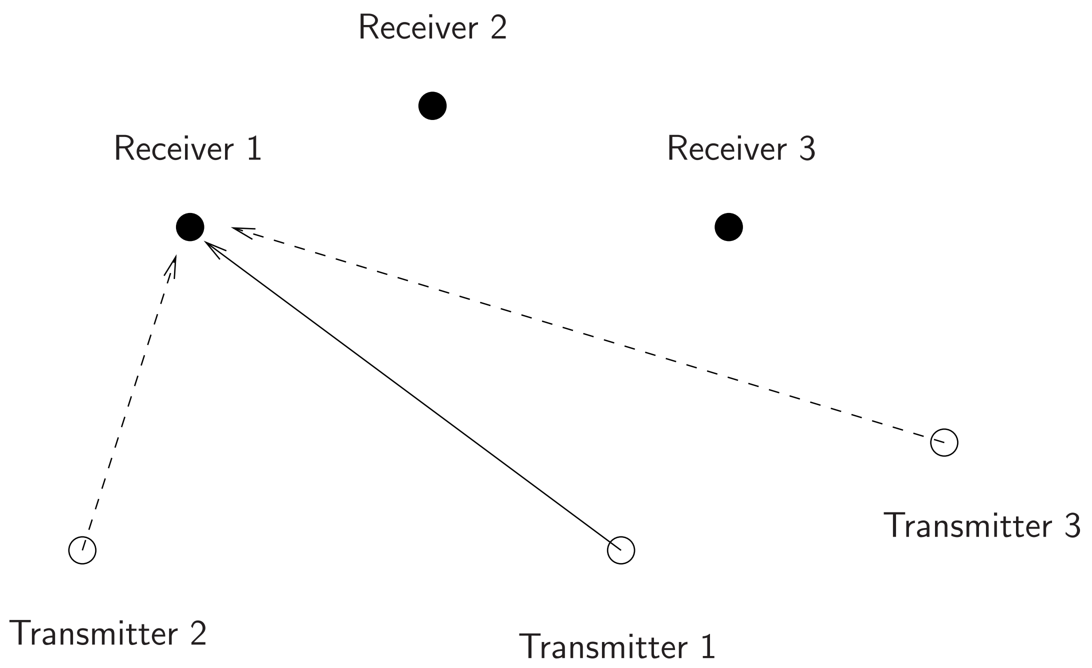
Generalized linear fractional program:

$$
\begin{aligned}
\min \quad & \max_{i=1,\ldots,K} \frac{c_i^T x + d}{f_i^T x + g_i} \\
\text{s.t.} \quad & Ax \preceq b \\
& f_i^T x + g_i > 0, \quad i = 1, \ldots, K
\end{aligned}
$$

- The objective function is quasiconvex.
- Can be solved using the bisection method.

Example: Optimal Power Assignment

- K transmitters, K receivers.



- Transmitter i sends signals to receiver i, & the other transmitters are interferers.
- The signal-to-interference-and-noise ratio (SINR) at receiver i

$$\gamma_i = \frac{G_{ii}p_i}{\sum_{j\neq i} G_{ij}p_j + \sigma_i^2}$$

where p_i is the transmitter i power, G_{ij} is the path gain from transmitter j to receiver i, and σ_i^2 is the noise power at receiver i.

- **Problem:** Maximize the smallest γ_i subject to power constraints $0 \leq p_i \leq p_{\mathsf{max},i}$, where $p_{\mathsf{max},i}$ is the max. allowable power of transmitter i.

  The power assignment problem is

$$\max_{\substack{p_i \in [0, p_{\max,i}] \\ i=1,\ldots,K}} \min_{i=1,\ldots,K} \frac{G_{ii} p_i}{\sum_{j \neq i} G_{ij} p_j + \sigma_i^2}$$

  which can be reformulated as a generalized linear fractional program:

$$\begin{aligned} \min \quad & \max_{i=1,\ldots,K} \frac{\sum_{j \neq i} G_{ij} p_j + \sigma_i^2}{G_{ii} p_i} \\ \text{s.t.} \quad & 0 \leq p_i \leq p_{\max,i}, \quad i = 1, \ldots, K \end{aligned}$$

- Note: The power assignment problem can alternatively be solved by geometric programming, or by a closed-form method that utilizes the problem structure $G_{ij} \geq 0$, $\sigma_i^2 > 0$.

Example: Another Optimal Power Assignment Problem

Problem: Minimize the average transmitter power, subject to a constraint that all SINRs are not less than a pre-specified threshold γ_o.

$$\begin{aligned} \min_{p} \ & \sum_{i=1}^{K} p_i \\ \text{s.t. } & \frac{G_{ii} p_i}{\sum_{j \neq i} G_{ij} p_j + \sigma_i^2} \geq \gamma_o, \quad i = 1, \ldots, K \\ & p_i \geq 0, \quad i = 1, \ldots, K \end{aligned}$$

- The problem can be rewritten as

$$
\begin{aligned}
\min_{p} \ & \sum_{i=1}^{K} p_i \\
\text{s.t.} \ & -G_{ii}p_i + \gamma_o \sum_{j \neq i} G_{ij} p_j + \gamma_o \sigma_i^2 \leq 0, \quad i = 1, \ldots, K \\
& p_i \geq 0, \quad i = 1, \ldots, K
\end{aligned}
$$

  which is an LP.

- Note: There is a closed-form solution to the LP above, when taking into account the problem structure $G_{ij} \geq 0$, $\sigma_i^2 > 0$.

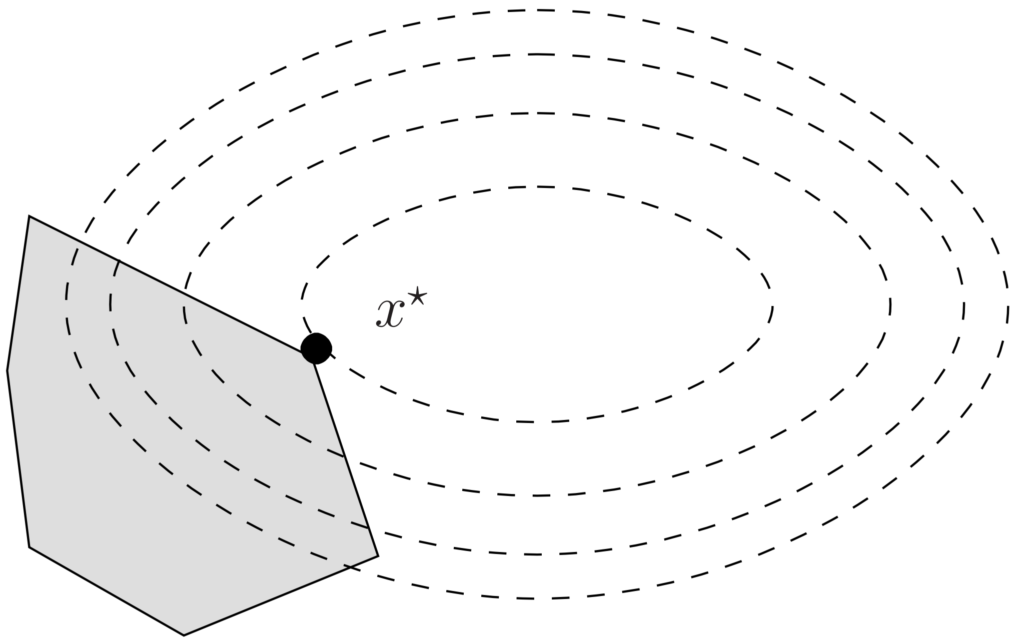
Additional Reading

[SB04] M. Schubert and H. Boche, "Solution of the multiuser downlink beamforming problem with individual SINR constraints," *IEEE Trans. Vehicular Tech.*, 2004.

Quadratic Programming (QP)

$$\begin{aligned} \min\ & \tfrac{1}{2}x^T P x + q^T x + r \\ \text{s.t.}\ & Ax = b, \quad Gx \preceq h \end{aligned}$$

A QP is convex iff $P \succeq 0$.



- Unconstrained QP (or LS)

$$\min \tfrac{1}{2}x^T P x + q^T x + r$$

  is a special case of QP where a closed form solution is available.

- The optimality condition is $Px = -q$.
  - If $P \succ 0$ then $x^\star = -P^{-1}q$.
  - If $P \succeq 0$ but $q \notin \mathcal{R}(P)$, then there is no solution for $Px = -q$. It can also be shown that $p^\star = -\infty$.
  - If $P \succeq 0$ & $q \in \mathcal{R}(P)$, then $x^\star = -P^\dagger q + \nu$ for any $\nu \in \mathcal{N}(P)$.

Examples:

- LS with bound constraints:

$$\begin{aligned} &\min \ \|Ax - b\|_2^2 \\ &\text{s.t. } \ell \preceq x \preceq u \end{aligned}$$

- Distance between between polyhedra:

$$\begin{aligned} &\min \ \|x_1 - x_2\|_2^2 \\ &\quad \text{s.t. } x_1 \in \{x | A_1 x \preceq b_1\}, \ x_2 \in \{x | A_2 x \preceq b_2\} \\ \Longleftrightarrow &\min \ \|x_1 - x_2\|_2^2 \\ &\quad \text{s.t. } A_1 x_1 \preceq b_1, \quad A_2 x_2 \preceq b_2 \end{aligned}$$

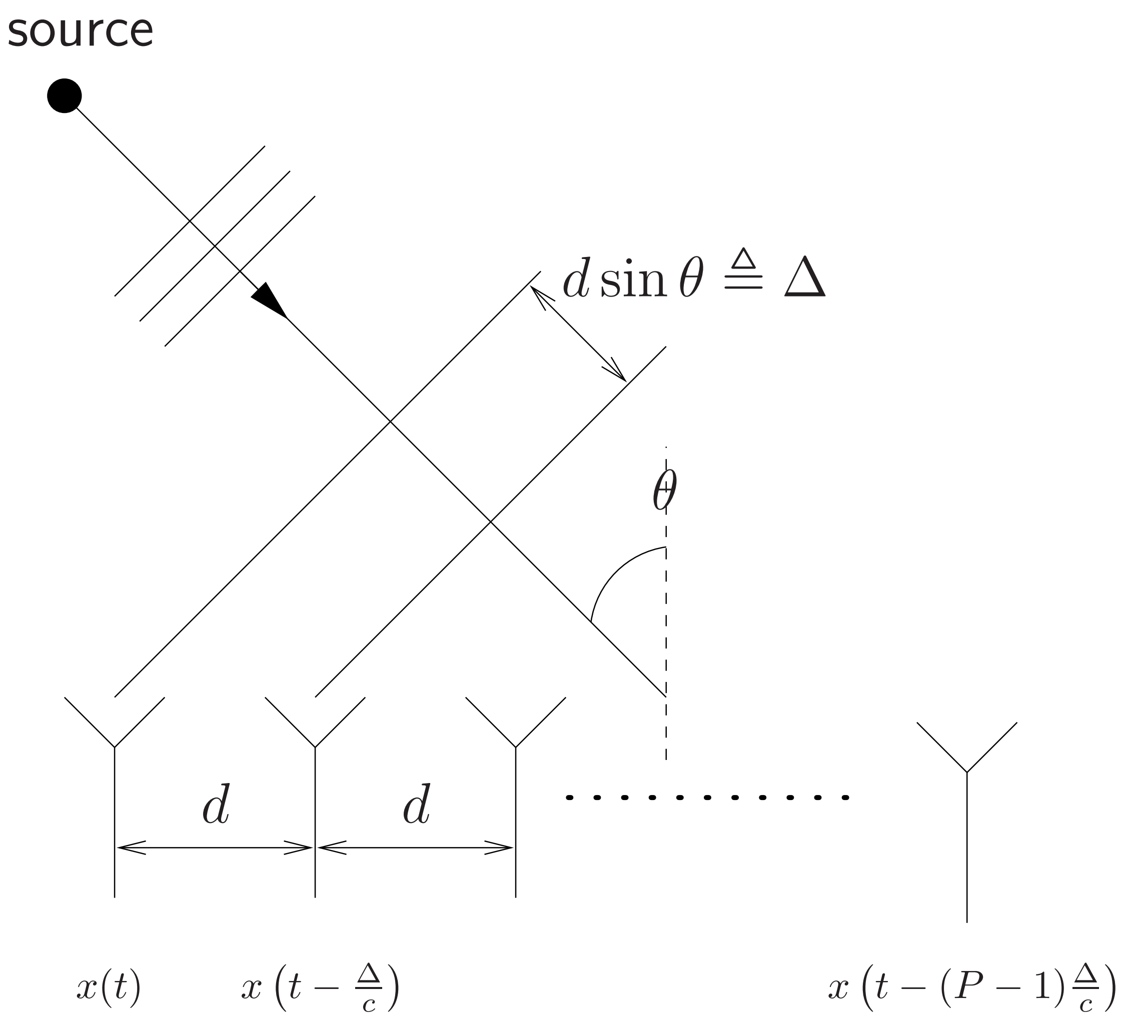
Quadratically Constrained QP (QCQP)

$$
\begin{aligned}
\min\ & \tfrac{1}{2}x^T P_0 x + q_0^T x + r_0 \\
\text{s.t.}\ & \tfrac{1}{2}x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{aligned}
$$

- QCQP is convex if $P_i \succeq 0$ for all i.
- When $P_i \succ 0$ for $i = 1, \ldots, m$, QCQP is a quadratic min. problem over an intersection of ellipsoids.
- If $P_i = 0$ for $i = 1, \ldots, m$, then QCQP reduces to QP.
- If $P_i = 0$ for $i = 0, 1, \ldots, m$, then QCQP reduces to LP.

Beamformer Design via QPs

Uniform linear array:



Signal model:

A1) far-field situations so that source waves are planar; &

A2) narrowband source signals so that the received signal of one sensor is a phase shifted version of that of another.

If a source signal $s(t) \in \mathbf{C}$ comes from a direction of θ, the array output $y(t) = [\ y_1(t), \ldots, y_P(t)\ ]^T$ is

$$y(t) = a(\theta)s(t)$$

Here,

$$a(\theta) = [\ 1, e^{-j2\pi d \sin(\theta)/\lambda}, \ldots, e^{-j2\pi d(P-1)\sin(\theta)/\lambda}\ ]^T \in \mathbf{C}^P$$

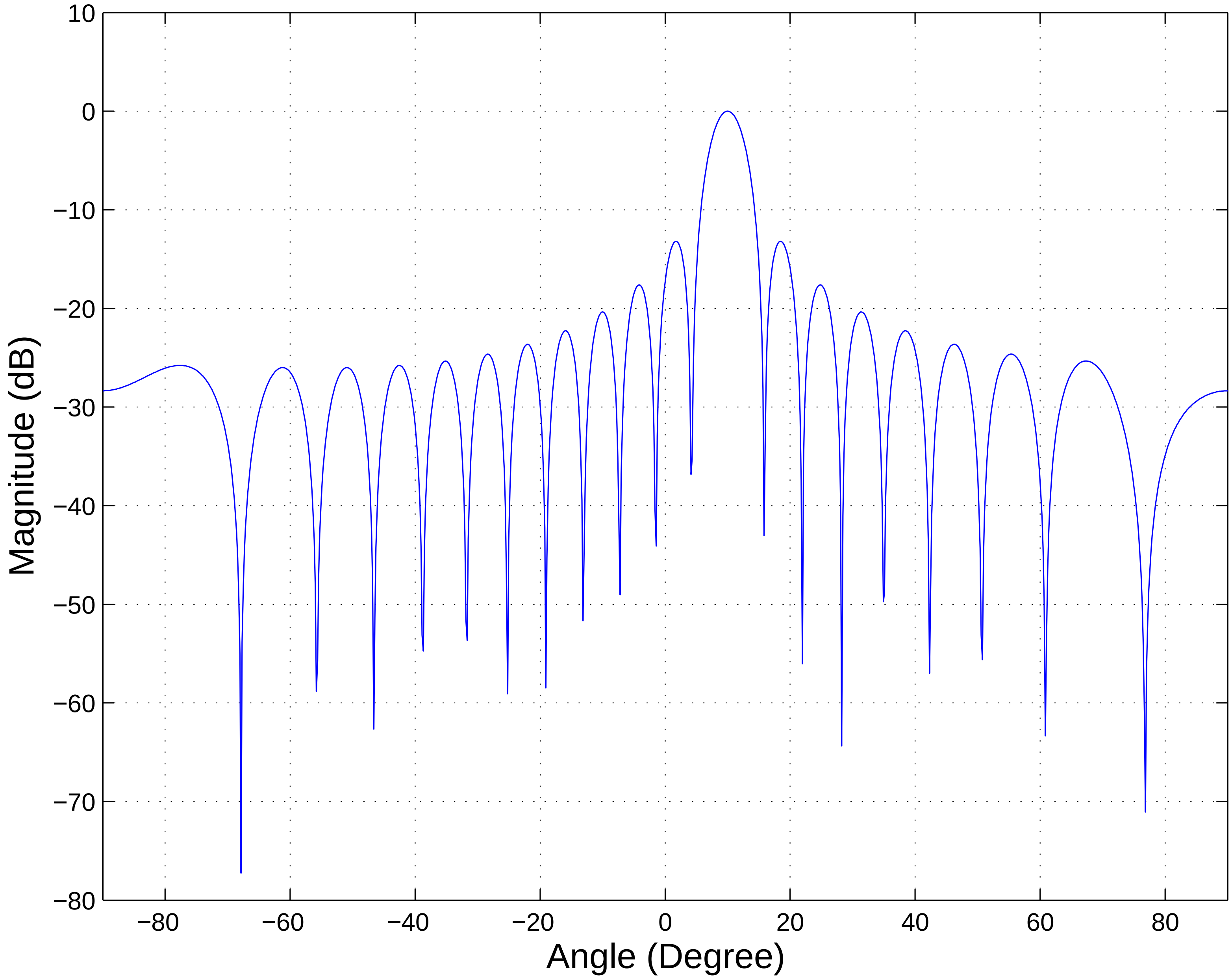
is the steering vector, where λ is the signal wavelength.

Beamforming:

$$\hat{s}(t) = w^H y(t)$$

where $w \in \mathbf{C}^P$ is a beamformer weight vector.

- Let $\theta_{\text{des}} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ be the desired direction.
- A simple beamformer is $w = a(\theta_{\text{des}})$, but it does not provide good sidelobe suppression.
- **Problem:** find a w that minimizes sidelobe energy subject to a pass response to θ_{des}.



Direction pattern of the conventional beamformer. $\theta_{\text{des}} = 10^\circ$; $P = 20$.

- Let $\Omega = [-\frac{\pi}{2}, \theta_\ell] \cup [\theta_u, \frac{\pi}{2}]$ denote the sidelobe band, for some θ_ℓ, θ_u so that $\theta_{\text{des}} \in [\theta_\ell, \theta_u]$.
- **Average sidelobe energy minimization:**

$$\begin{aligned} \min \; & \int_\Omega |w^H a(\theta)|^2 d\theta \\ \text{s.t.} \; & w^H a(\theta_{\text{des}}) = 1 \end{aligned}$$

The problem is equiv. to an equality constrained QP:

$$\begin{aligned} \min \; & w^H P w \\ \text{s.t.} \; & w^H a(\theta_{\text{des}}) = 1 \end{aligned}$$

where $P = \int_\Omega a(\theta) a^H(\theta) d\theta$ (can be computed by numerical integration).

- **Worst-case sidelobe energy minimization:**

$$\begin{aligned} \min \ & \max_{\theta \in \Omega} |w^H a(\theta)|^2 \\ \text{s.t. } & w^H a(\theta_{\text{des}}) = 1 \end{aligned}$$

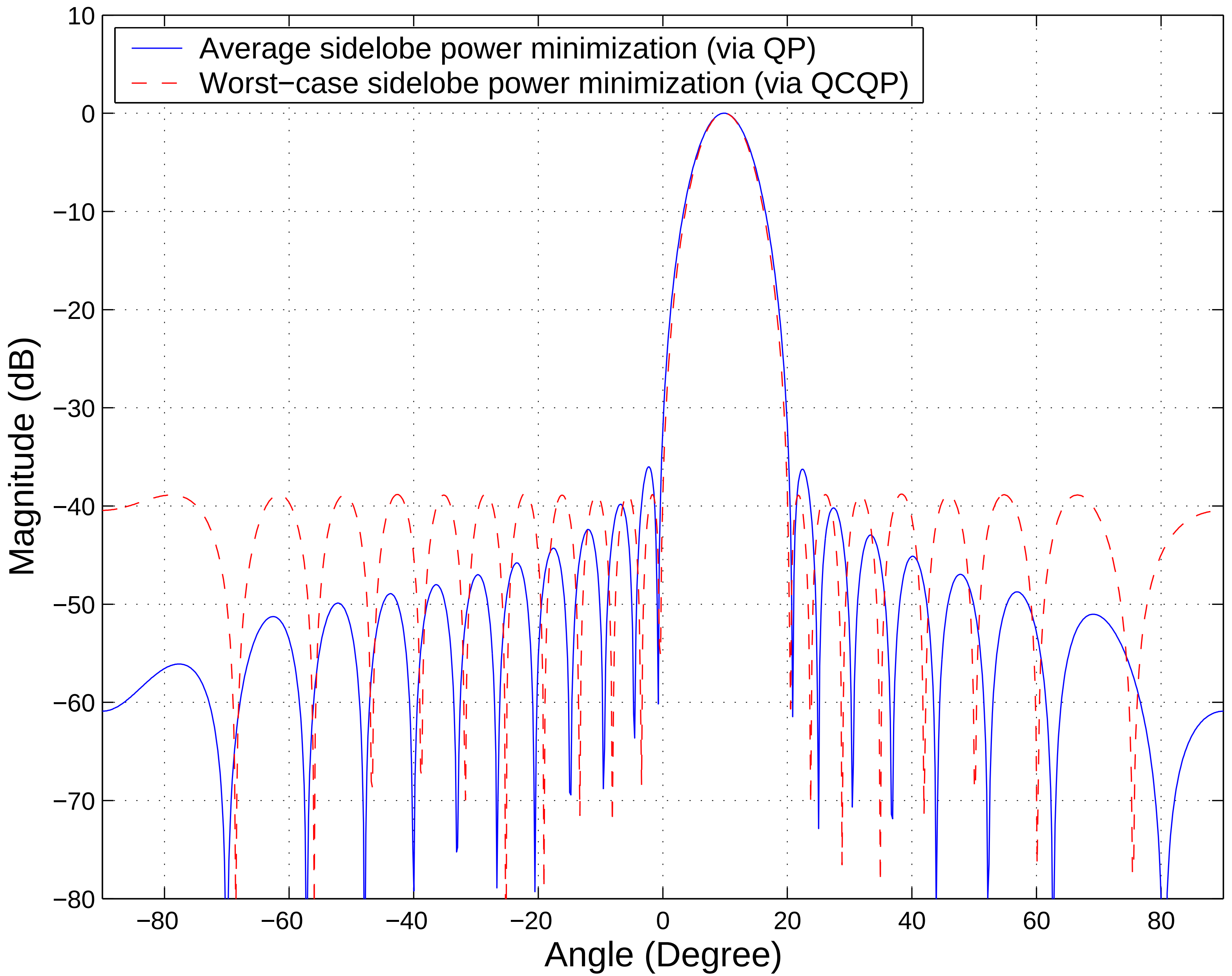
The problem can be reformulated as

$$\begin{aligned} \min \ & t \\ \text{s.t. } & |w^H a(\theta)|^2 \leq t, \quad \theta \in \Omega \\ & w^H a(\theta_{\text{des}}) = 1 \end{aligned}$$

which is a QCQP with semi-infinite constraints.

- The worst-case sidelobe energy minimization problem can be approximated by discretization.
- Let $\theta_1, \theta_2, \cdots, \theta_L$ be some set of sample points in Ω. We approximate the problem by

$$
\begin{aligned}
&\min\ t \\
&\text{s.t. } |w^H a(\theta_i)|^2 \leq t, \quad i = 1, \ldots, L \\
&\qquad w^H a(\theta_{\text{des}}) = 1
\end{aligned}
$$



Direction patterns of the two beamformer designs. $\theta_{\text{des}} = 10^\circ$. Sidelobe suppression is applied to directions outside $[0^\circ, 20^\circ]$.