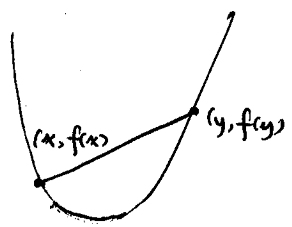
March 19, 2006

III. CONVEX FUNCTIONS



Definition: A function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if

i) dom f is convex

ii) for all $x, y \in \text{dom} f$, $\theta \in [0,1]$,

$$f(\theta x_1 + (1-\theta) y) \le \theta f(x_1) + (1-\theta) f(y).$$

- The set $S = \{x \mid f(x) \le a\}$ is convex if f is convex:

Let $x_1, x_2 \in S$. Since, for any $\theta \in [0,1]$,

$$\begin{aligned} f(\theta x_1 + (1-\theta)x_2) &\le \theta f(x_1) + (1-\theta) f(x_2) \\ &\le a \end{aligned}$$

we have $\theta x_1 + (1-\theta)x_2 \in S$.

- f is convex iff it is convex when restricted to any line that intersects its domain; i.e., $\forall\ x \in \text{dom} f$ and v, $g(t) = f(x+tv)$ is convex over $\{t \mid x + tv \in \text{dom} f\}$.

Necessity: For $\theta \in [0,1]$, & $t_1, t_2 \in \text{dom}\, g$,

$$\begin{aligned} g(\theta t_1 + (1-\theta)t_2) &= f(\theta(x+t_1 v) + (1-\theta)(x + t_2 v)) \\ &\le \theta f(x+t_1 v) + (1-\theta) f(x+t_2 v) \\ &= \theta g(t_1) + (1-\theta) g(t_2). \end{aligned}$$

thus showing g is convex.

Sufficiency: For each $x_1, x_2 \in \mathbb{R}^n$, $\theta \in [0,1]$, let

$$x = x_1, \qquad v = x_2 - x_1.$$

Since g is convex,

$$g(\theta t_1 + (1-\theta) t_2) \le \theta g(t_1) + (1-\theta) g(t_2).$$

Now, let $t_1 = 0$, and $t_2 = 1$. The above eqn. becomes

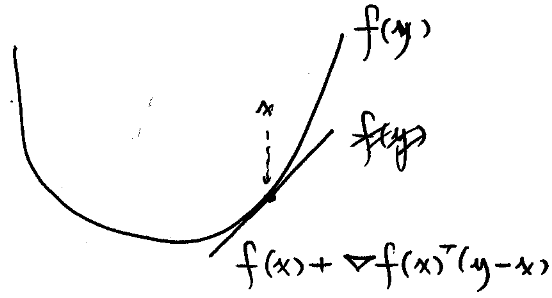
$$\begin{aligned} f(x_1 + (1-\theta)(x_2 - x_1)) &= f(\theta x_1 + (1-\theta) x_2) \\ &\le \theta f(x) + (1-\theta) f(x+v) \\ &= \theta f(x_1) + (1-\theta) f(x_2). \end{aligned}$$

- 1st order condition: Suppose f is differentiable (∇f exists at each pt. in $\mathrm{dom}\, f$, ~~which is open~~) & $\mathrm{dom}\, f$ is open

f is convex iff $\mathrm{dom}\, f$ is convex and

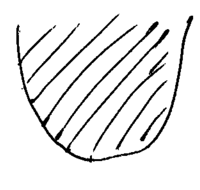
$$f(y) \ge f(x) + \nabla f(x)^T (y - x)$$

holds for all $x, y \in \mathrm{dom}\, f$.



Remind us of the supporting hyperplane, particularly when we look at

$$\{x \mid f(x) \le a\}$$



~~Proof of the 1st order condition:~~

(First reminder of Taylor series)

$$f(x_0+u) = f(x_0) + df(x_0) + \frac{1}{2!} d^2 f(x_0) + \cdots + \frac{1}{(n-1)!} d^{n-1} f(x_0) + R_n$$

where

$$d^r f(x_0) = \underbrace{\sum_i \sum_j \cdots \sum_k}_{r \text{ sums}} u_i u_j \cdots u_k \frac{\partial f}{\partial x_i \partial x_j \cdots \partial x_k}\bigg|_{x=x_0},$$

$$R_n = \frac{1}{n!} d^n f(x_0 + \theta u), \quad \text{for some } \theta \in (0,1)$$

Proof of 1st order condition:

Necessity:

$$f(x+\lambda(y-x)) = f(x) + \lambda \nabla f(x+\theta\lambda(y-x))^T (y-x)$$

Since f is convex,

$$f(x+\lambda(y-x)) = f(\lambda y + (1-\lambda)x)$$
$$\leq \lambda f(y) + (1-\lambda) f(x).$$

Thus

$$\lambda f(y) \geq \lambda f(x) + \lambda \nabla f(x+\theta\lambda(y-x))^T(y-x).$$

For $\lambda > 0$, $\lambda \to 0$,

$$f(y) \geq f(x) + \nabla f(x)^T (y-x).$$

Sufficiency: We have, for any $y, z \in \mathrm{dom}\, f$,

$$f(y) \geq f(x) + \nabla f(x)^T (y-x)$$
$$f(z) \geq f(x) + \nabla f(x)^T (z-x)$$

Thus,

$$\lambda f(y) + (1-\lambda) f(z) \geq f(x) + \nabla f(x)^T (\lambda y + (1-\lambda) z - x)$$

Set $x = \lambda y + (1-\lambda) f(z)$. Then we have

$$\lambda f(y) + (1-\lambda) f(z) \geq f(\lambda y + (1-\lambda) f(z)).$$

- 2nd order condition: Suppose f is twice differentiable.
  f is convex iff its Hessian is PSD:

$$\nabla^2 f(x) \succeq 0, \quad \forall x \in \text{dom} f$$

- f is strictly convex if $\mathrm{dom}\, f$ is convex, and

$$f(\theta x + (1-\theta) y) < \theta f(x) + (1-\theta) f(y)$$

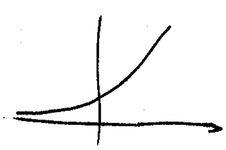
whenever $x, y \in \mathrm{dom}\, f$, $x \neq y$, $0 < \theta < 1$. f is strictly convex ~~iff~~ if $\nabla^2 f(x) \succ 0$. (converse not true)

- f is concave if $-f$ is convex. f is concave iff $\nabla^2 f(x) \preceq 0$.

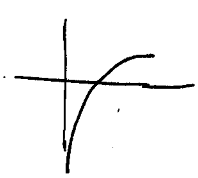
- Examples

  - e^{ax} is convex on $\mathbb{R}$:

$$\nabla f = a e^{ax}$$
$$\nabla^2 f = a^2 e^{ax} \geq 0$$



  - $\log x$ is concave on $\mathbb{R}_{++}$.



$$\nabla f = 1/x$$
$$\nabla^2 f = -1/x^2 < 0, \quad x \in \mathbb{R}_{++}$$

  - $x \log x$ is convex on $\mathbb{R}_{++}$.

$$\nabla f = 1 + \log x$$
$$\nabla^2 f = 1/x > 0, \quad x \in \mathbb{R}_{++}$$

- $\log \int_{-\infty}^{x} e^{-t^2/2}\,dt$ is concave (prove it!)

- Some more examples:

- $f(x) = a^T x + b$ (affine function)

  is both convex and concave.

- $f(x) = x^T P x + 2q^T x + r$ (quadratic function) is convex iff $P \succeq 0$. ($\because \nabla^2 f(x) = P$).

- Every norm on $\mathbb{R}^n$ $f(x) = \|x\|$ is convex. $\because$

$$\|\theta x + (1-\theta) y\| \le \theta \|x\| + (1-\theta)\|y\|$$

- $f(x) = \max\{x_1, \cdots, x_n\}$. is convex.

$$\begin{aligned} f(\theta x + (1-\theta) y) &= \max_i \left(\theta x_i + (1-\theta) y_i\right) \\ &\le \theta \max_i x_i + (1-\theta) \max_i y_i \\ &= \theta f(x) + (1-\theta) f(y). \end{aligned}$$

- Geometric mean $f(x) = \left(\prod_{i=1}^{n} x_i \right)^{1/n}$ is concave on $\mathbb{R}^n_{++}$.

- ~~log-exp-sum-exp~~ log-sum-exp $f(x) = \log \left(\sum_{i=1}^{n} e^{x_i} \right)$ is convex on $\mathbb{R}^n$.

  Note that

$$\max\{x_1, \cdots, x_n\} \le f(x) \le \log n + \max\{x_1, \cdots, x_n\}.$$

  Convexity proven by checking Hessian.

$$\log\left(\sum_{i=1}^{n} e^{x_i}\right) \ge \log e^{x_i} = x_i, \text{ for any } i$$

- Some more examples for matrix functions:

  - $f(X) = \mathrm{tr}(AX)$ is linear.

  - $f(X) = \log\det X$. is concave on S^n_{++}.

  < Provide 'line' property right here.

  f is concave iff

$$h(t) = -\log\det(X + tV)$$

  is convex on $\{t \mid X + tV \succ 0\}$ for any $X \in S^n$, $V \in S^n$.

  w.l.o.g. assume $X \succ 0$.

$$
\begin{aligned}
h(t) &= -\log\det\left(X^{1/2} (I + tX^{-1/2} V X^{-1/2}) X^{1/2} \right) \\
&= -\log\det X^{1/2} \det(I + tX^{-1/2} V X^{-1/2}) \det X^{1/2} \\
&= -\log\det X - \log\det(I + tX^{-1/2} V X^{-1/2}).
\end{aligned}
$$

  Let $\lambda_1, \cdots, \lambda_n$ denote the (real) eigenvalues of $X^{-1/2} V X^{-1/2}$.

$$
\begin{aligned}
h(t) &= -\log\det X - \log \prod_{i=1}^{n} (1 + t\lambda_i) \\
&= -\log\det X - \sum_{i=1}^{n} \log(1 + t\lambda_i)
\end{aligned}
$$

$$h'(t) = -\sum_i \frac{1}{1+t\lambda_i}\lambda_i$$

$$h''(t) = \cancel{\sum_i + \frac{1}{(1+t\lambda_i)^2}\lambda_i^2 \mp \frac{1}{1+t\lambda_i}} \sum_i \frac{\lambda_i^2}{(1+t\lambda_i)^2} \geq 0.$$

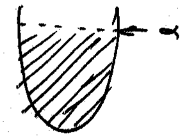
$$\cancel{= \sum_i \frac{+\lambda_i^2 \mp 1 \mp t\lambda_i}{(1+t\lambda_i)^2}}$$

Hence, h is convex implying that f is concave.

- Sublevel sets

$$C_\alpha = \{ x \in \operatorname{dom} f \mid f(x) \le \alpha \}$$

C_α is convex for any α, if f convex.



- Epigraph

$$\{ (x, f(x)) \mid x \in \operatorname{dom} f \} \subseteq \mathbb{R}^{n+1}$$

is the graph of the function f.

$$\operatorname{epi} f = \{ (x,t) \mid x \in \operatorname{dom} f,\ f(x) \le t \}$$

is the epigraph of f.

- f is convex iff $\operatorname{epi} f$ is convex.

proof: necessity (f convex $\Rightarrow$ $\operatorname{epi} f$ convex)

Let $(x_1, t_1), (x_2, t_2) \in \operatorname{epi} f$. Since f is convex

$$f(\theta x_1 + (1-\theta) x_2) \le \theta f(x_1) + (1-\theta) f(x_2), \quad \theta \in [0,1]$$
$$\le \theta t_1 \quad + (1-\theta) t_2$$

$$\Rightarrow \theta (x_1, t_1) + (1-\theta)(x_2, t_2) \in \operatorname{epi} f.$$

Sufficiency ($\operatorname{epi} f$ convex $\Rightarrow$ f convex)

For any $(x_1, t_1), (x_2, t_2) \in \operatorname{epi} f$,

$$f(\theta x_1 + (1-\theta) x_2) \le \theta t_1 + (1-\theta) t_2.$$

Choose $t_1 = f(x_1)$ and $t_2 = f(x_2)$. Then we have f being convex.

- Jensen's Inequality:

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$$

for convex f and $\theta \in [0,1]$.

Let $p(x) \geq 0$ on $S \subseteq \operatorname{dom} f$, and $\int_S p(x)\,dx = 1$.

$$f\left(\int_S x\,p(x)\,dx\right) \le \int_S f(x)\,p(x)\,dx$$

$$f(E\{x\}) \quad \le \quad E\{f(x)\}.$$

- Example: arithmetic-geometric mean inequality:

$$\sqrt{ab} \le \left(\frac{a+b}{2}\right)$$

for $a, b \geq 0$.

The function $-\log x$ is convex on $\mathbb{R}_{++}$.

$$-\log(\theta a + (1-\theta)b) \le -\theta \log a - (1-\theta)\log b.$$

or

$$\log(\theta a + (1-\theta)b) \ge \cancel{\theta} \log a^{\theta} + \log b^{(1-\theta)} = \log a^{\theta} b^{(1-\theta)}$$

$$\theta a + (1-\theta)b \ge a^{\theta} b^{(1-\theta)}.$$

for $a, b > 0$. (For $a=0$ or $b=0$ the inequality is trivial).

Set $\theta = 1/2$, we obtain arithmetic-geometric inequality.

CONVEXITY PRESERVING OPERATIONS

- NONNEGATIVE WEIGHTED SUMS

$$\left.\begin{array}{l} f_1, \cdots, f_m \text{ convex} \\ w_1, \cdots, w_m \geq 0 \end{array}\right. \Rightarrow \sum_{i=1}^{m} w_i f_i \text{ convex}$$

proof:

$$\sum_{i=1}^{m} w_i f_i(\theta x + (1-\theta) y) \leq \sum_{i=1}^{m} w_i \left(\theta f_i(x) + (1-\theta) f_i(y)\right)$$

$$= \theta \sum_{i=1}^{m} w_i f_i(x) + (1-\theta) \sum_{i=1}^{m} f_i(y)$$

extension to integrals.

$$\left.\begin{array}{l} f(x,y) \text{ is convex in } x \\ \text{for each } y \in \mathcal{A}, \\ w(y) \geq 0 \text{ for each } y \in \mathcal{A} \end{array}\right. \Rightarrow g(x) = \int_{\mathcal{A}} w(y) f(x,y)\, dy \text{ convex}$$

- COMPOSITION WITH AN AFFINE MAPPING

$$g(x) = f(Ax + b)$$

is convex if f is convex.

- POINTWISE MAX. & SUPREMUM

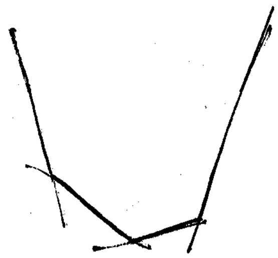
f_1, f_2 convex $\Rightarrow f(x) = \max\{f_1(x), f_2(x)\}$ convex.

proof:

$$f(\theta x + (1-\theta)y) = \max\{f_1(\theta x+(1-\theta)y), f_2(\theta x+(1-\theta)y)\}$$
$$\leq \max\{\theta f_1(x) + (1-\theta)f_1(y), \theta f_2(x) + (1-\theta)f_2(y)\}$$
$$\leq \theta \max\{f_1(x), f_2(x)\} + (1-\theta)\max\{f_1(y), f_2(y)\}.$$

Example: Piecewise-linear function:

$$f(x) = \max_{i=1,\dots,L} a_i^T x + b_i$$



Extension:

$f(x,y)$ is convex in x for each $y \in \mathcal{A}$ $\Rightarrow$ $g(x) = \sup_{y \in \mathcal{A}} f(x,y)$ convex

proof: $$\operatorname{epi} g = \{(x,t) \mid \sup_{y\in\mathcal{A}} f(x,y) \le t\}$$
$$= \{(x,t) \mid f(x,y) \le t,\ \forall y \in \mathcal{A}\}$$
$$= \bigcap_{y\in\mathcal{A}} \{(x,t) \mid f(x,y) \le t\}$$
$$= \bigcap_{y\in\mathcal{A}} \operatorname{epi} f(\cdot, y)$$

$$\underset{\text{convex}}{\cancel{\text{epi}}\, f(\cdot,y)} \Leftrightarrow \underset{\text{convex}}{\operatorname{epi} f(\cdot,y)} \Rightarrow \underset{\text{convex}}{\operatorname{epi} g} \Leftrightarrow \underset{\text{convex}}{g} \quad !$$

Example: $f(x) = \sup_{y\in C} \|x - y\|$ is convex (note that C is even not convex!)

Example: $$f(X) = \lambda_{max}(X)$$
$$= \sup\{y^T X y \mid \|y\|_2 = 1\}$$
$$= \sup_{\|y\|_2=1} \operatorname{tr}(X y y^T)$$ is convex.

Example: Norm of $X \in \mathbb{R}^{n\times m}$

$$f(X) = \sup\{u^T X v \mid \|u\|_2 = 1,\ \|v\|_2 = 1\}$$

$\sup\{\|Xu\|_2 \mid \|u\|_2 = 1\}$

$$= \sup_{\substack{\|u\|_2=1,\\ \|v\|_2=1}} \operatorname{tr}(X v u^T)$$

is convex.

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◦ COMPOSITION

Let $f(x) = h(g(x))$, $h: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R}^n \to \mathbb{R}$. Define an extended-value function of h as follows: If h is convex

$$\tilde{h}(x) = \begin{cases} h(x), & x \in \operatorname{dom} h \\ \infty, & x \notin \operatorname{dom} h \end{cases}$$

If h is concave

$$\tilde{h}(x) = \begin{cases} h(x), & x \in \operatorname{dom} h \\ -\infty, & x \notin \operatorname{dom} h. \end{cases}$$

Then,

f is convex if h is convex, $\tilde{h}$ is nondecreasing, & g is convex.

f is convex if h is convex, $\tilde{h}$ is nonincreasing, & g is concave

f is concave if h is concave, $\tilde{h}$ is nondecreasing, & g is concave

f is concave if h is concave, $\tilde{h}$ is nonincreasing, & g is convex.

EXAMPLE: Consider $n=1$, and both g and h are differentiable.

$$f'(x) = h'(g(x))\, g'(x)$$
$$f''(x) = h''(g(x))\,(g'(x))^2 + h'(g(x))\, g''(x).$$

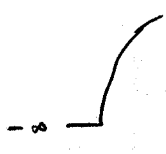
If g and h are convex, then $g''(x) \geq 0$ and $h''(x) \geq 0$. $\forall x$.

If h is nondecreasing, then $h'(x) \geq 0$ $\forall x$.

The consequence is that $f''(x) \geq 0$ $\forall x$.

EXAMPLE:

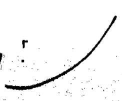
$h(x) = \log x$, $\operatorname{dom} h = \mathbb{R}_{++}$ is concave and $\tilde{h}$ is nondecreasing.



$h(x) = x^{1/2}$, $\operatorname{dom} h = \mathbb{R}_+$, is concave & $\tilde{h}$ is nondecreasing.

BUT $h(x) = x^2$, $\operatorname{dom} h = \mathbb{R}_+$, is convex but $\tilde{h}$ is not nondecreasing!

$h(x) =$ ~~$\max\{0, x^2\}$~~ $(\max\{0, x\})^2$ is okay!



- Minimization

$$\left.\begin{array}{l} f(x,y) \text{ convex in } (x,y) \\ C \text{ convex nonempty} \end{array}\right\} \Rightarrow \quad g(x) = \inf_{y\in C} f(x,y)$$

convex,

provided $g(x) > -\infty$ for ~~all~~ ~~some~~ some x.

Let $x_1, x_2 \in \operatorname{dom} g = \{x \mid (x,y) \in \operatorname{dom} f \text{ for some } y \in C\}$.

Let $\epsilon > 0$. There are $y_1, y_2 \in C$ such that $f(x_i, y_i) \le g(x_i) + \epsilon$ for $i = 1, 2$.

$$\begin{aligned} g(\theta x_1 + (1-\theta)x_2) &= \inf_{y\in C} f(\theta x_1 + (1-\theta)x_2, y) \\ &\le f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) \\ &\le \theta f(x_1, y_1) + (1-\theta) f(x_2, y_2) \\ &\le \theta g(x_1) + (1-\theta) g(x_2) + \epsilon \end{aligned}$$

Since this holds for any $\epsilon > 0$ (or for any $\epsilon > 0$, we can find y_1, y_2),

$$g(\theta x_1 + (1-\theta)x_2) \le \theta g(x_1) + (1-\theta) g(x_2)$$

Alternate Proof for Minimization: Suppose $\exists\, y$ s.t. $f(x,y) = \inf\{ - - - \}$. p.16 plus

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$$\mathrm{epi}\, g = \{ (x,t) \mid f(x,y) \le t \text{ for some } y \in C \}$$

Let

$$\mathcal{H} = \{ (x,y,t) \mid f(x,y) \le t,\ y \in C \}.$$

It can be shown that $\mathcal{H}$ is convex. Since

$$\mathrm{epi}\, g = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathcal{H}$$

is ~~an affine mapping of~~ an image of an affine mapping,

epi g is convex.

Example: Schur complement.

$$f(x,y) = x^T A x + 2x^T B y + y^T C y$$
$$= [x^T \ y^T] \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

where $A, \cancel{B} \in S^n$, $B \in S^m$.

Suppose f is convex, which holds iff $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0$.

Consider
$$g(x) = \inf_{y \in \mathbb{R}^m} f(x,y)$$
$$= f(x^*, y^*), \quad y^* = -C^+ B^T x$$
$$= x^T A x - 2x^T B C^+ B^T x + x^T B C^+ B^T x$$
$$= x^T (A - BC^+B^T) x.$$

Since g is convex, it must hold true that

$$A - BC^+B^T \succeq 0$$

Example: $\mathrm{dist}(x,S) = \inf_{y \in S} \|x-y\|$ is convex for convex S.

- PERSPECTIVE OF A FUNCTION

The perspective of a function f is a function $g: \mathbb{R}^{n+1} \to \mathbb{R}$

$$g(x,t) = t f(x/t), \quad \mathrm{dom}\, g = \{(x,t) \mid x/t \in \mathrm{dom} f,\ t > 0\}$$

If f is convex, then g is convex.

proof: for $t > 0$

$$(x,t,s) \in \mathrm{epi}\, g \iff t f(x/t) \leq s$$
$$\iff f(x/t) \leq s/t$$
$$\iff (x/t, s/t) \in \mathrm{epi}\, f$$

Apparently, g is the inverse image of epi f under the perspective mapping (convex set lecture). So g is convex for convex f.

Example: $f(x) = x^T x$.

$$g(x,t) = t \frac{x^T x}{t^2} = \frac{1}{t} x^T x$$

is convex in (x,t) for $t > 0$.

Example: $f(x) = -\log x$ is convex on $\mathbb{R}_{++}$.

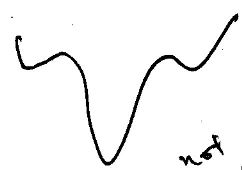
$$g(x,t) = -t \log x/t = t \log t/x$$

is convex on $\mathbb{R}^2_{++}$. As an aside, g is the relative entropy of t and x.

$$g(u,v) = \sum_{i=1}^{n} u_i \log(u_i/v_i)$$

is convex on $\mathbb{R}^{2n}_{++}$.

• QUASICONVEX FUNCTIONS

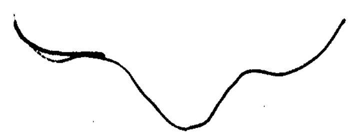


not quasiconvex

$f: \mathbb{R}^n \to \mathbb{R}$ is quasiconvex (or unimodal) if

$$S_\alpha = \{ x \in \mathrm{dom}\, f \mid f(x) \le \alpha \}$$

is convex for every α.

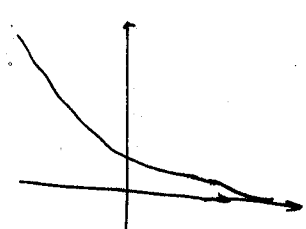


quasiconvex

f is quasiconcave if $-f$ is quasiconvex.

f is quasilinear if f is quasiconvex and quasiconcave.

If $\{x \mid f(x) = \alpha\}$ is convex, then f is quasilinear.



Quasilinear

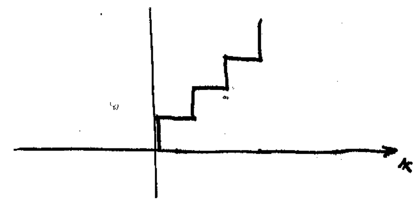
EXAMPLE: $f(x) = \log x$ is concave on $\mathbb{R}_{++}$. Now

$$S_\alpha = \{x \mid \log x \le \alpha\} = \{x \mid x \le e^\alpha\}$$

is convex. Hence f is quasiconvex. It is also quasiconcave.

EXAMPLE: Ceiling function $f(x) = \inf\{z \in \mathbb{Z} \mid z \ge x\}$.

$$S_\alpha = \{x \mid \inf\{z \in \mathbb{Z} \mid z \ge x\} \le \alpha\}$$
$$= \{x \mid x \le \alpha\}$$



Hence, the ceiling function is quasiconvex (also q.-concave).

EXAMPLE: Linear fractional function

$$f(x) = \frac{a^T x + b}{c^T x + d}, \quad \operatorname{dom} f = \{x \mid c^T x + d > 0\}$$

$$S_\alpha = \{x \mid c^T x + d > 0,\ a^T x + b \le \alpha(c^T x + d)\}$$

is a polyhedron, and thus f is q.-convex.
Likewise we can show that f is q.-concave.

Modified

- Jensen inequality for q.-convex functions

A function f is q.-convex iff for any $x, y \in \mathrm{dom} f$, and $0 \le \theta \le 1$,

$$f(\theta x + (1-\theta) y) \le \max\{f(x), f(y)\}$$

proof: necessity:

Let x, y ~~$\in S_\alpha$~~. Choose $\alpha = \max\{f(x), f(y)\}$. Then, $x, y \in S_\alpha$.
Since S_α is convex, for $0 \le \theta \le 1$

$$\theta x + (1-\theta) y \in S_\alpha$$

$$\Longleftrightarrow \theta x + (1-\theta) y \le \alpha = \max\{f(x), f(y)\}.$$

Sufficiency:

For every α, pick two points $x, y \in S_\alpha$. Since

$$f(\theta x + (1-\theta) y) \le \max\{f(x), f(y)\} \le \alpha,$$

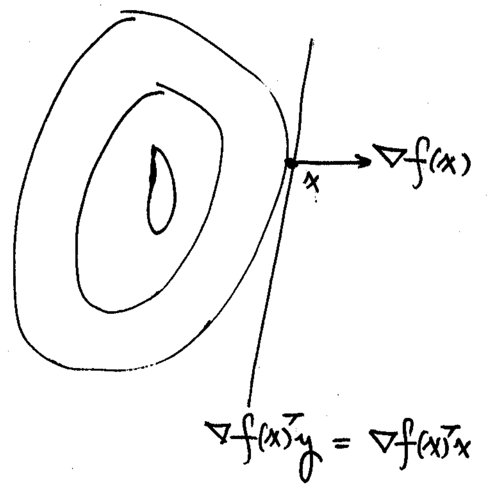
we have $\theta x + (1-\theta) y \in S_\alpha$. This implies S_α is convex.

~~FIRST-ORDER~~ CONDITIONS FOR Q-CONVEX FUNCTIONS

- Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable.

  f is q-convex iff $\operatorname{dom} f$ is convex and $\forall\ x, y \in \operatorname{dom} f$

$$f(y) \leq f(x) \Rightarrow \nabla f(x)^T (y-x) \leq 0.$$



Essentially $\{y \mid \nabla f(x)^T(y-x) \leq 0\}$ provides a supporting hyperplane for S_α, $\alpha = f(x)$.

- Suppose f is twice differentiable.

$$f \text{ quasiconvex} \Longrightarrow \forall\ x, \cancel{y} \in \operatorname{dom} f,\ y \in \mathbb{R}^n$$

$$y^T \nabla f(x) = 0 \Rightarrow y^T \nabla^2 f(x)\, y \geq 0$$

Let's look at ~~dom~~ $\operatorname{dom} f = \mathbb{R}$. The condition becomes

$$f'(x) = 0 \Rightarrow f''(x) \geq 0.$$

In general, we have 2 cases:

i) $\nabla f(x) = 0$: in that case $\nabla^2 f(x) \succeq 0$.

ii) $\nabla f(x) \neq 0$: $y \in R_\perp(\nabla f(x))$ and $y^T \nabla^2 f(x)\, y \geq 0$ for those y.

This means $\nabla^2 f(x)$ can have 1 -ve eigenvalue whilst the rest must be non-ve.

OPERATIONS THAT PRESERVE Q.-CONVEXITY

- Nonnegative weighted max.:

$$f = \max\{ w_1 f_1, \ldots, w_m f_m \}$$

is q.-convex if f_i are q.-convex and $w_i \geq 0$.

- Composition:

If $g: \mathbb{R}^n \to \mathbb{R}$ is q.-convex and $h: \mathbb{R} \to \mathbb{R}$ is nondecreasing, then $h(g(x))$ is q.-convex.

- Minimization:

If $f(x,y)$ is q.-convex in (x,y) and C is a convex set,

$$g(x) = \inf_{y \in C} f(x,y)$$

is q.-convex.

CONVEXITY W.R.T. GENERALIZED INEQUALITY

- MONOTONICITY W.R.T. GENERALIZED INEQUALITY

A function $f: \mathbb{R}^n \to \mathbb{R}$ is K-nondecreasing if

$$x \preceq_K y \Rightarrow f(x) \le f(y).$$

and K-increasing if

$$\begin{matrix} x \preceq_K y \\ x \neq y \end{matrix} \Rightarrow f(x) < f(y)$$

EXAMPLE: Consider $K = S^n_+$.

- $tr(WX)$ is K-nondecreasing if $W \succeq 0$.

For $X \preceq Y$, $tr(W(Y-X)) \ge 0 \Rightarrow tr(WY) \ge tr(WX)$.

Likewise, $tr(WX)$ is K-nonincreasing if $W \preceq 0$.

EXAMPLE: ~~$K = S^n_{++}$ (note: S^n_{++} is not proper!)~~

$tr(X^{-1})$ is K-decreasing on S^n_{++}.

• Suppose f is differentiable.

f, with a convex domain, is K-nondecreasing iff

$$\nabla f(x) \succeq_{K^*} 0, \quad \forall x \in \mathrm{dom} f$$

and K-increasing if

$$\nabla f(x) \succ_{K^*} 0.$$

CONVEXITY W.R.T. GENERALIZED INEQUALITY

$f: \mathbb{R}^n \to \mathbb{R}^m$ is K-convex if for all $x, y \in \mathrm{dom} f$, and $0 \le \theta \le 1$,

$$f(\theta x + (1-\theta) y) \preceq_K \theta f(x) + (1-\theta) f(y)$$

and strictly K-convex if

$$f(\theta x + (1-\theta) y) \prec_K \theta f(x) + (1-\theta) f(y)$$

for all $x \neq y$, and $0 < \theta < 1$.

- For $K = \mathbb{R}^n_+$, f is a function where each component function $f_i(x)$ is convex.