Convex Sets

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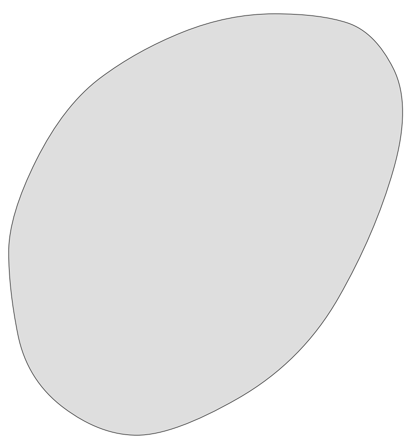
Convex Sets

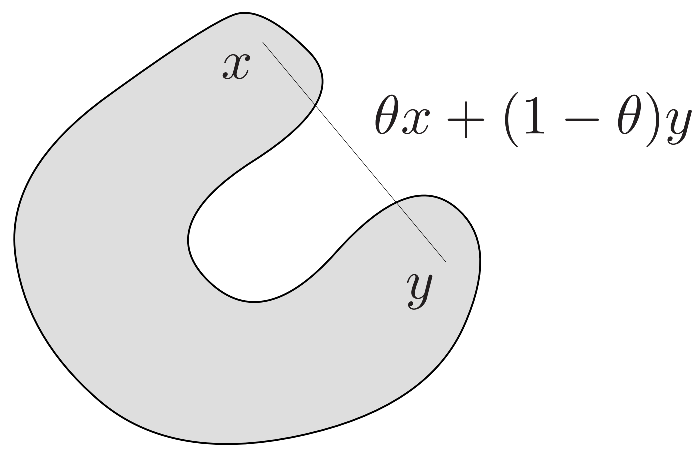
A set $C \subseteq \mathbf{R}^n$ is said to be **convex** if, for any $x, y \in C$,

$$\theta x + (1 - \theta) y \in C$$

for any $0 \leq \theta \leq 1$.



convex



non-convex

- The line segment of any two points in C has to be in C, in order to be convex.

Examples of Convex Sets

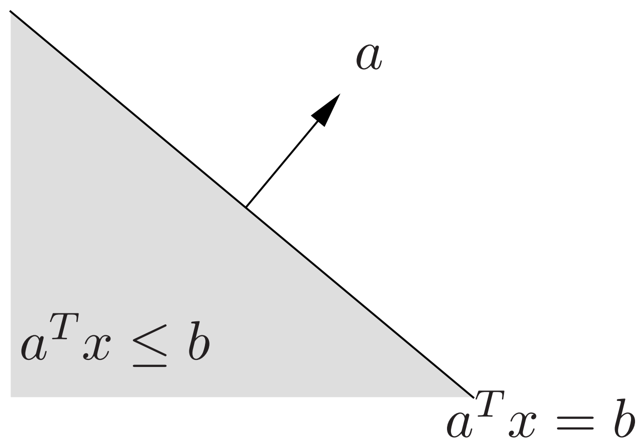
Hyperplane:

$$C = \{x \mid a^T x = b\}$$

where $a \in \mathbf{R}^n$, & $b \in \mathbf{R}$.

Halfspace:

$$C = \{x \mid a^T x \leq b\}$$



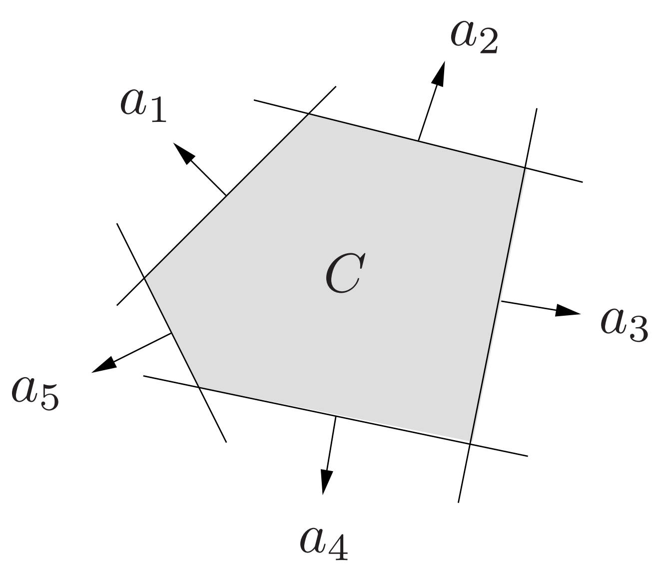
Polyhedron:

$$C = \{x \mid a_i^T x \leq b_i, i = 1, \ldots, m,\ c_i^T x = d_i, i = 1, \ldots, p\}$$

For convenience we use matrix notations to represent a polyhedron:

$$C = \{x \mid Ax \preceq b,\ Cx = d\}$$

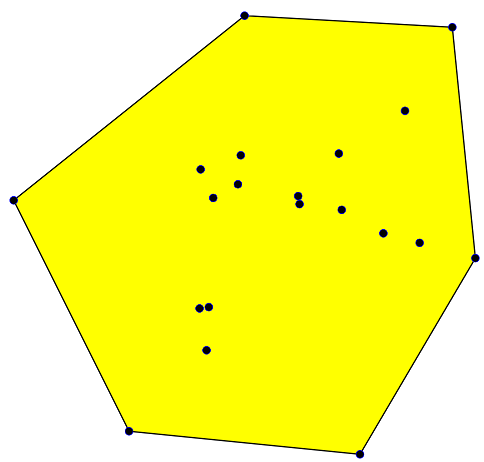
where $A \in \mathbf{R}^{m \times n}$, $C \in \mathbf{R}^{p \times n}$, $b \in \mathbf{R}^m$, $d \in \mathbf{R}^p$, & $\preceq$ denotes elementwise inequality.



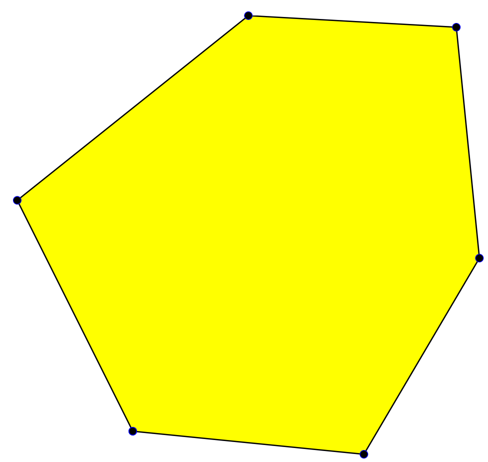
Convex hull of a set of points $\{x_1, x_2, \ldots, x_k\}$:

$$C = \text{conv}\{x_1, x_2, \ldots, x_k\} = \{x = \theta_1 x_1 + \ldots + \theta_k x_k \mid \theta_1 + \ldots + \theta_k = 1,\ \theta_1, \ldots, \theta_k \geq 0\}$$

- The set of all convex combinations of $\{x_1, x_2, \ldots, x_k\}$
- $\text{conv}\{x_1, \ldots, x_k\}$ is a polyhedron. (vice versa is true if polyhedron is bounded)



(a) Convex hull where only some of the $x_1, x_2, \ldots, x_k$ are vertices.



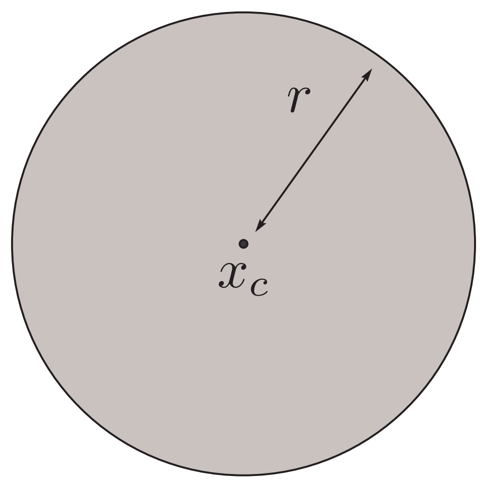
(b) Convex hull where all $x_1, \ldots, x_k$ are vertices.

- $x \in C$ is an **extreme point** or **vertex** of C if $x \neq \sum_{i=1}^{k} \theta_i x_i$ for any $\theta_1 + \ldots + \theta_k = 1$, $\theta_1, \ldots, \theta_k \geq 0$, $\theta_i \neq 1$ for any i.

Euclidean ball:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\}$$

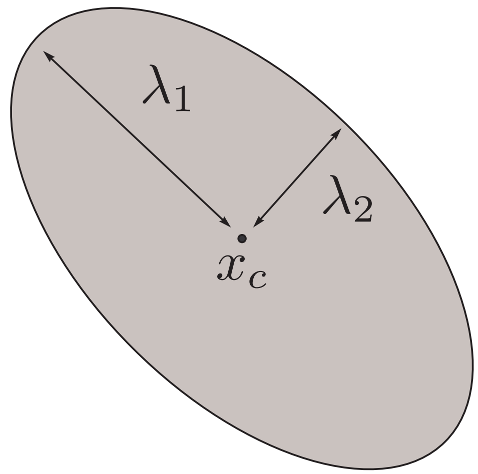
$$= \left\{ x \;\middle|\; \sqrt{\sum_{i=1}^{n} (x_i - x_{c,i})^2} \leq r \right\}$$



Ellipsoid:

$$\mathcal{E}(x_c, P) = \{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\}$$

where $P \in \mathbf{S}^n$ ($\mathbf{S}^n$ = the set of $n \times n$ symmetric matrices), and P is positive semidefinite.



Symmetric eigendecomposition of P:

$$P = Q \Lambda Q^T$$

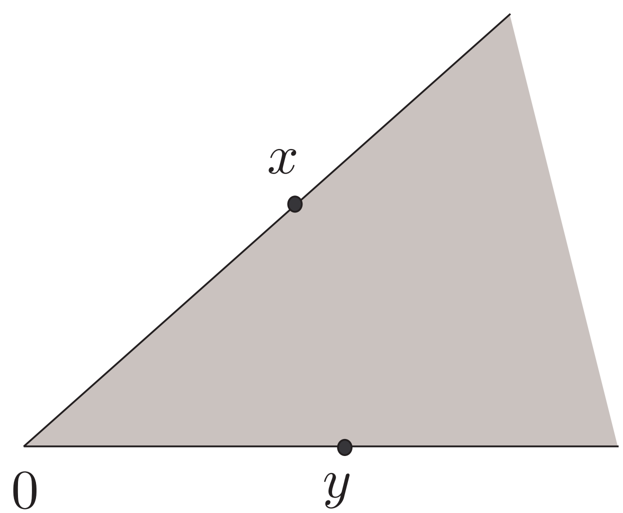
- The eigenvector matrix Q ($Q \in \mathbf{R}^{n \times n}$, $Q^T Q = I$) controls the rotation;
- The eigenvalue matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ controls the lengths of the semi-axes.

Convex Cones

A set $C \subseteq \mathbf{R}^n$ is said to be a **convex cone** if, for any $x, y \in C$,

$$\theta_1 x + \theta_2 y \in C$$

for any $\theta_1, \theta_2 \geq 0$.

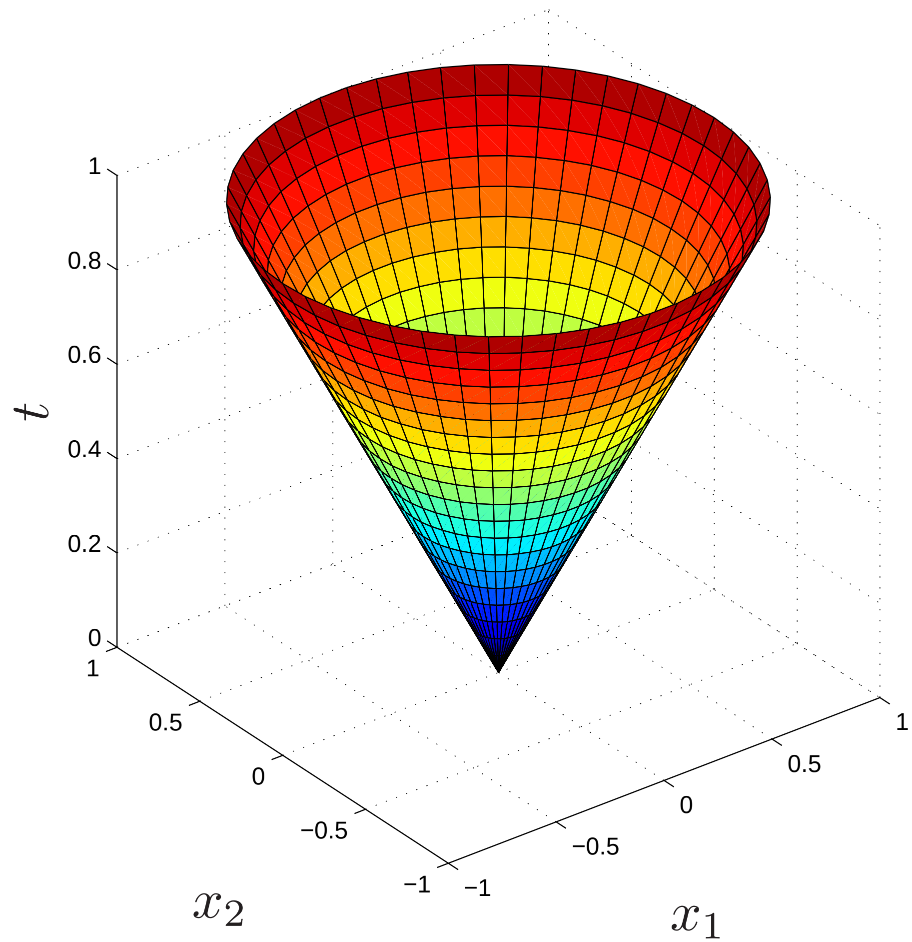


- A convex cone is a convex set.

Examples of Convex Cones

Second-order cone (SOC) (aka Lorentz cone, or ice-cream cone):

$$K = \{(x,t) \mid \|x\|_2 \le t\}$$



Positive semidefinite (PSD) cone:

$$\mathbf{S}_+^n = \{X \in \mathbf{S}^n \mid X \succeq 0\}$$

where $X \succeq 0$ means that X is PSD; i.e.,

$$z^T X z \geq 0, \quad \text{for all } z \in \mathbf{R}^n$$

Positive definite (PD) cone:

$$\mathbf{S}_{++}^n = \{X \in \mathbf{S}^n \mid X \succ 0\}$$

where $X \succ 0$ means that X is PD; i.e.,

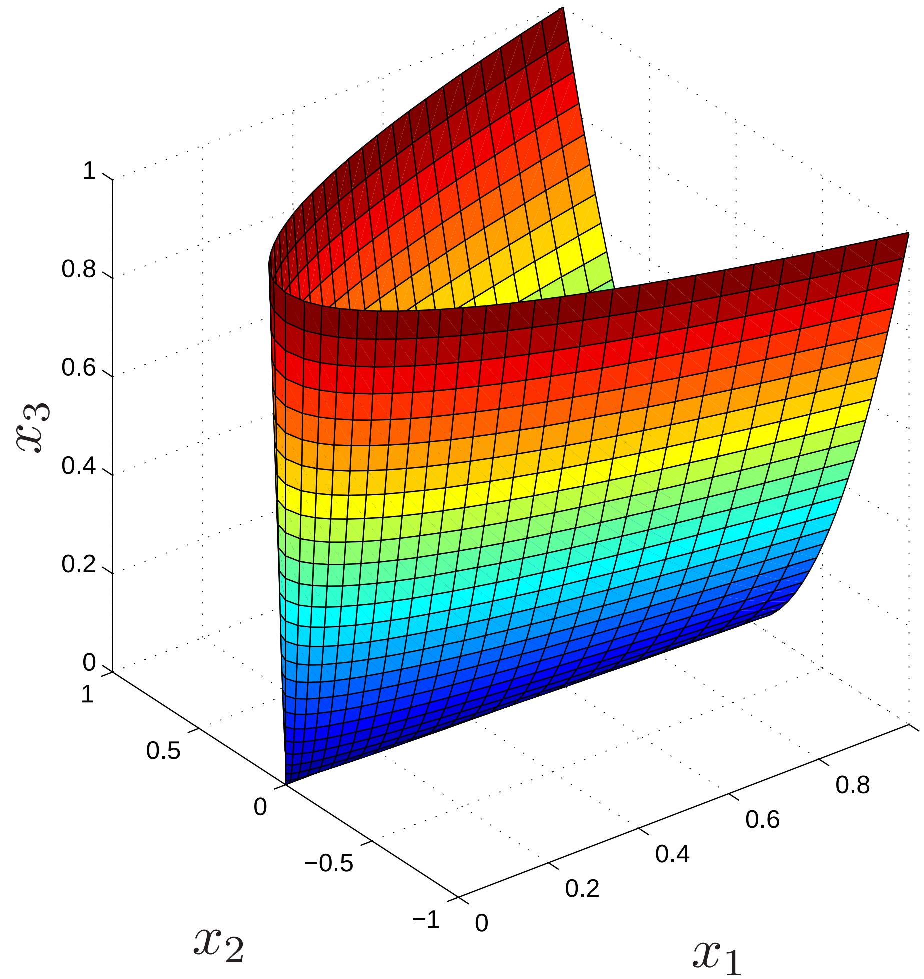
$$z^T X z > 0, \quad \text{for all } z \in \mathbf{R}^n/\{0\}$$

Example: $n = 2$

$$\begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \succeq 0$$

$$\Longleftrightarrow x_1 x_3 - x_2^2 \geq 0,$$

$$x_1 \geq 0, x_3 \geq 0$$



Convexity Preserving Operations

Intersection:

$$S_1, S_2, \ldots S_k \text{ convex} \Longleftrightarrow S_1 \cap S_2 \cap \ldots \cap S_k \text{ convex}$$

Example: Polyhedron

$$C = \{x \mid a_i^T x \leq b_i, i = 1, \ldots, m,\ c_i^T x = d_i, i = 1, \ldots, p\}$$

is convex, because it is an intersection of halfspaces $\{x \mid a_i^T x \leq b_i\}$ and hyperplanes $\{x \mid c_i^T x = d_i\}$.

Extension of Intersection:

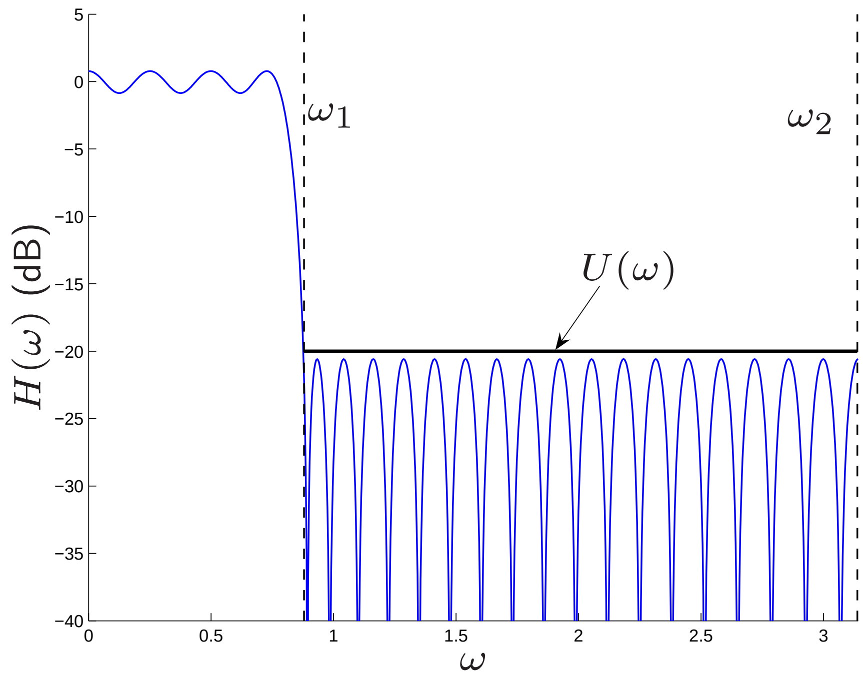
$$S_\alpha \text{ convex for every } \alpha \in \mathcal{A} \Longleftrightarrow \bigcap_{\alpha \in \mathcal{A}} S_\alpha \text{ convex}$$

Example: Filter mask

- Let $\{h_{-n}, \ldots, h_{-1}, h_0, h_1 \ldots, h_n\}$ be a set of FIR filter coefficients. Assume $h_{-i} = h_i$ (linear phase).

- The frequency response

$$H(\omega) = \sum_{i=-n}^{n} h_i e^{-j\omega i}$$
$$= h_0 + 2\sum_{i=1}^{n} h_i \cos(\omega i)$$



- The set

$$\mathcal{H} = \left\{(h_0, \ldots, h_n) \in \mathbf{R}^{n+1} \;\middle|\; |H(\omega)| \leq U(\omega),\ \omega_1 \leq \omega \leq \omega_2\right\}$$

where $U(\omega) \geq 0$, is convex because

$$\mathcal{H} = \bigcap_{\omega_1 \leq \omega \leq \omega_2} \underbrace{\left\{(h_0, \ldots, h_n) \in \mathbf{R}^{n+1} \;\middle|\; -U(\omega) \leq H(\omega) \leq U(\omega)\right\}}_{\text{polyhedral for each } \omega}$$

Affine function: Suppose $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is affine; i.e.,

$$f(x) = Ax + b$$

where $A \in \mathbf{R}^{m \times n}$, and $b \in \mathbf{R}^m$.

$$S \text{ convex} \Longrightarrow f(S) = \{f(x) \mid x \in S\} \text{ convex}$$
$$C \text{ convex} \Longrightarrow f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

Example: $\{(y,t) \in \mathbf{R}^{m+1} \mid \|y\|_2 \leq t\}$ is convex, so

$$\{x \in \mathbf{R}^n \mid \|Ax + b\|_2 \leq c^T x + d\}$$

is convex.

Example: Set of linear matrix inequalities (LMI)

$$\{x \in \mathbf{R}^n \mid A_0 + x_1 A_1 + \ldots + x_n A_n \preceq 0\}$$

where $A_i \in \mathbf{S}^m$, is convex, since it is an inverse image of $\mathbf{S}^m_+ = \{Y \mid Y \succeq 0\}$.

Generalized Inequalities

A convex cone K is a **proper cone** if

- K is closed (has boundary)
- K is solid (has nonempty interior)
- K is pointed ($x \in K, -x \in K \Longrightarrow x = 0$)

(Important) examples:

- nonnegative orthant $K = \mathbf{R}_+^n = \{x \in \mathbf{R}^n \mid x_i \geq 0, i = 1, \ldots, n\}$
- SOC $K = \{(x, t) \in \mathbf{R}^{n+1} \mid \|x\|_2 \leq t\}$
- PSD cone $K = \{X \in \mathbf{S}^n \mid X \succeq 0\}$

Generalized inequality defined by a proper cone K:

$$x \preceq_K y \Longleftrightarrow y - x \in K$$

$$x \prec_K y \Longleftrightarrow y - x \in \mathbf{int} K \quad \text{(strict ineq.)}$$

For example, for $K = \mathbf{R}^n_+$,

$$x \preceq_K y \Longleftrightarrow x_i \leq y_i, \quad i = 1, \ldots, n$$

Properties of $\preceq_K$ are generally the same as those of $\leq$ on $\mathbf{R}$; e.g.,

$$x \preceq_K y,\ y \preceq_K z \quad \Longrightarrow \quad x \preceq_K z$$

$$x \preceq_K 0,\ x \succeq_k 0 \quad \Longrightarrow \quad x = 0$$

Schur Complement

- Consider

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

  and assume $C \succ 0$.

- $A - BC^{-1}B^T$ is called the **Schur complement.** Important property:

$$X \succeq 0 \Longleftrightarrow A - BC^{-1}B^T \succeq 0$$

Example:

$$C = \{(X, x) \in \mathbf{S}^n \times \mathbf{R}^n \mid \underbrace{X \succeq xx^T}_{\text{or } X - xx^T \text{ PSD}} \}$$

By Schur complement, C is equivalent to

$$C = \left\{ (X, x) \in \mathbf{S}^n \times \mathbf{R}^n \;\middle|\; \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0 \right\}$$

- Schur complement is a useful tool. It shows that a no. of sets can be turned to an LMI sets.

Example: the Euclidean ball $C = \{x \in \mathbf{R}^n \mid \|x - x_c\|_2 \leq 1\}$.

$$
\begin{aligned}
C &= \{x \mid (x - x_c)^T (x - x_c) \leq 1\} \\
&= \{x \mid 1 - (x - x_c)^T I (x - x_c) \geq 0\} \\
&= \left\{ x \;\middle|\; \begin{bmatrix} 1 & (x - x_c)^T \\ x - x_c & I \end{bmatrix} \succeq 0 \right\} \quad \text{(Schur complement)}
\end{aligned}
$$

Example: An SOC $C = \{(x, t) \in \mathbf{R}^{n+1} \mid \|x\|_2 \leq t,\ t > 0\}$.

$$
\begin{aligned}
C &= \{(x, t) \mid \|x\|_2^2 \leq t^2,\ t > 0\} \\
&= \{(x, t) \mid t - x^T (\tfrac{1}{t} I) x \geq 0,\ t > 0\} \\
&= \left\{ x \;\middle|\; \begin{bmatrix} t & x^T \\ x & tI \end{bmatrix} \succeq 0,\ t > 0 \right\} \quad \text{(Schur complement)}
\end{aligned}
$$