II. Convex Sets – Pt 1.

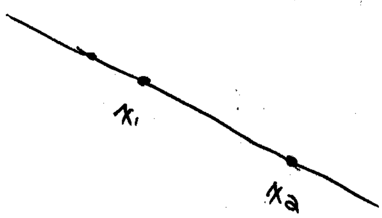
Affine Sets

Line:

- Suppose $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$.

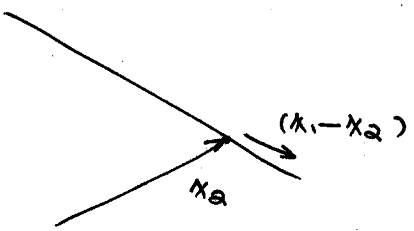
$$y = \theta x_1 + (1-\theta)x_2$$

where $\theta \in \mathbb{R}$ form the line passing through x_1 & x_2



Note that

$$y = x_2 + \theta(x_1 - x_2)$$



where x_2 is the base,

$(x_1 - \cancel{y}\, x_2)$ is the direction.

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- A set $C \subseteq \mathbb{R}^n$ is affine if

$$x_1, x_2 \in C \Rightarrow \theta x_1 + (1-\theta)x_2 \in C \text{ for every } \theta \in \mathbb{R}.$$

- A point

$$y = \sum_{i=1}^{k} \theta_i x_i, \quad \sum_{i=1}^{k} \theta_i = 1$$

is an affine combination of the points $x_1, \cdots, x_k$.

- Using induction, we can show that:

If C is affine, $x_1, \cdots, x_k \in C$, $\theta_1 + \cdots + \theta_k = 1$

then the affine combination $\sum_{i=1}^{k} \theta_i x_i \in C$.

- If C is affine & $x_0 \in C$ (any pt. in C can serve as a base),

$$V = C - x_0 = \{ x - x_0 \mid x \in C \}$$

is a subspace.

Proof:

Suppose $v_1, v_2 \in V$. Then $v_1 + x_0,\ v_2 + x_0 \in C$.

For $\alpha, \beta \in \mathbb{R}$,

$$\alpha v_1 + \beta v_2 + x_0 = \alpha(v_1 + x_0) + \beta(v_2 + x_0) + (1 - \alpha - \beta)x_0 \in C.$$

This implies

$$\alpha v_1 + \beta v_2 \in C - x_0 = V.$$

- Thus, an affine set can be expressed as

$$\begin{aligned} C &= V + x_0 \\ &= \{ Az + x_0 \mid z \in \mathbb{R}^r \} \end{aligned}$$

for some $A \in \mathbb{R}^{n \times r}$ so that $V = \mathcal{R}(A)$.

- The dim. of an affine set is the dim. of V.

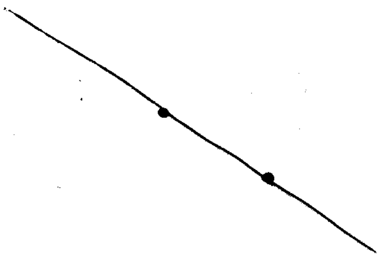
- Affine hull

$$\mathrm{aff}\, C = \left\{ \theta_1 x_1 + \cdots + \theta_k x_k \;\middle|\; x_1, \dots, x_k \in C,\ \sum_{i=1}^{k} \theta_i = 1 \right\}$$

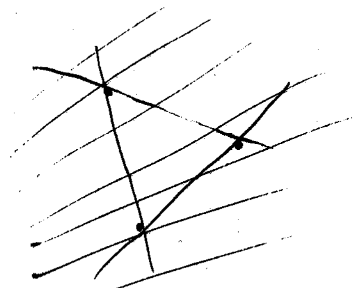
Note that C is not necessarily affine, but $\mathrm{aff}\, C$ is so.

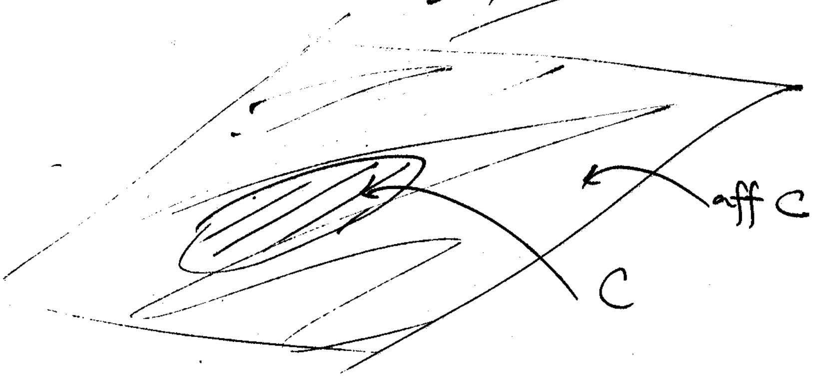
e.g.

$C = \{x_1, x_2\}$ (finite set) $\qquad \mathrm{aff}\, C = \{ \theta x_1 + (1-\theta) x_2 \mid \theta \in \mathbb{R} \}$



$C = \{x_1, x_2, x_3\} \subseteq \mathbb{R}^2 \qquad \mathrm{aff}\, C = \mathbb{R}^3$





- The affine hull is the smallest affine set that contains C, i.e. If S is any affine set with $C \subseteq S$, then $\mathrm{aff}\, C \subseteq S$.

Proof:

Suppose $\mathrm{aff}\, C \supset S$.

Then there are $x_1, \cdots, x_k \in C$, $\theta_1 + \cdots + \theta_k = 1$ such that

$$y = \sum_{i=1}^{k} \theta_i x_i \in \mathrm{aff}\, C$$

and yet $y \notin S$.

Now $C \subseteq S$ implies $x_1, \cdots, x_k \in C$. As an affine set y must lie in S. This is contradictory.

- The affine dim. of C is the dim. of its affine hull.

dimension of an affine set is the dim. of its subspace counterpart; i.e., $\dim C = \dim(C - x_0)$, $x_0 \in C$.

e.g., $C = \{ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1, x_3 = 0 \}$.

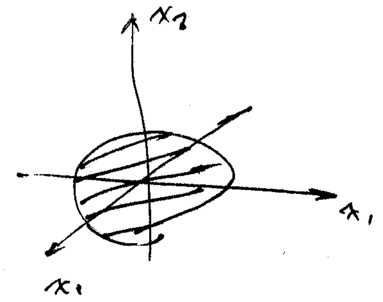
$\mathrm{aff}\, C = \{ x \in \mathbb{R}^3 \mid x_3 = 0 \}$

$\Rightarrow$ The aff. dim. of C is 2.

- The <u>relative interior</u> of C is the interior relative to $\mathrm{aff}\, C$:

$$\mathrm{relint}\, C = \{ x \in C \mid B(x, r) \cap \mathrm{aff}\, C \subseteq C \text{ for some } r > 0 \}.$$

e.g.. $C = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq 1,\ x_3 = 0\}$



There is NO interior pt. of C, coz any norm ball $B(x, r)$ is 3 dimensional!
(C is not open ∄, but should be closed).

$\Leftrightarrow$ cl $C = C$! bd $C =$ cl $C = C$!

Now aff $C = \{x \in \mathbb{R}^3 \mid x_3 = 0\}$.

relint $C = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < 1,\ x_3 = 0\}$

- The relative boundary of C is cl $C \setminus$ relint C.

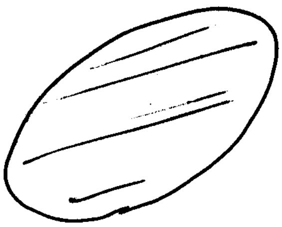
In the above example the relative boundary is $\{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1\}$.

Convex Sets

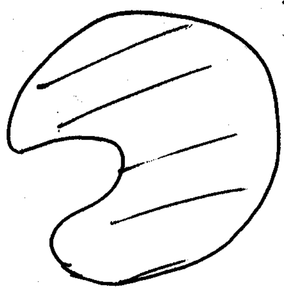
- A set C is convex if

$$x_1, x_2 \in C \Rightarrow \theta x_1 + (1-\theta)x_2 \in C \quad \forall \; \theta \in [0,1]$$

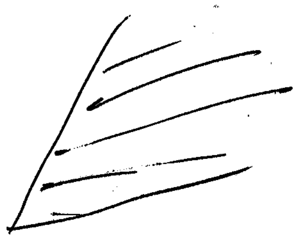
Examples:



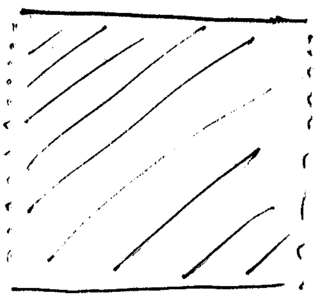
convex



nonconvex



convex



nonconvex

$I_1 = [a,b]$, $I_2 = [c,d]$

$I_1 \cap I_2 = \emptyset$.

$I_1 \cup I_2$ nonconvex.

- $y = \theta_1 x_1 + \cdots + \theta_k x_k$, where $\theta_i \geq 0 \; \forall \, i$, $\sum_{i=1}^{k} \theta_i = 1$

is a convex combination of the pts $x_1, \ldots, x_k$.

- convex hull of a set C

$$\text{conv } C = \left\{ \theta_1 x_1 + \cdots + \theta_k x_k \;\middle|\; x_i \in C,\ \theta_i \geq 0,\ i=1,\ldots,k \quad \theta_1 + \cdots + \theta_k = 1 \right\}$$

- generalization to infinite sums and integrals:

$$\left.\begin{array}{l} \theta_i \geq 0, \quad i = 1, 2, \ldots \\ \sum_{i=1}^{\infty} \theta_i = 1 \\ x_1, x_2, \ldots \in C, \ C \text{ convex} \end{array}\right. \Rightarrow \sum_{i=1}^{\infty} \theta_i x_i \in C$$

if the series converges

$$\left.\begin{array}{l} p: \mathbb{R}^n \to \mathbb{R} \text{ satisfies } p(x) \geq 0 \\ \text{for all } x \in C \\ \quad (\int_C p(x)\,dx = 1.) \\ C \subseteq \mathbb{R}^n \text{ is convex} \end{array}\right. \Rightarrow \int_C p(x)\, x \, dx \in C$$

if the integral exists.

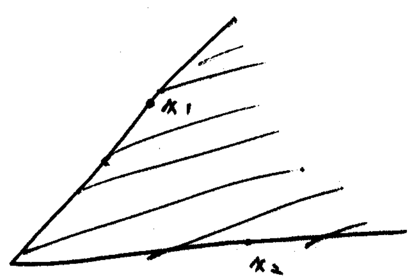
May be thought as $p(x)$ being a prob. density, ~~and~~ ~~$\{\theta_i\}$ being prob. distribution (for discrete variables)~~.

- Cones

- A set C is a cone, or non-ve homogeneous if for every $x \in C$ and $\theta \geq 0$ we have $\theta x \in C$.

- A set C is a convex cone if

$$\left.\begin{array}{l} x_1, x_2 \in C \\ \theta_1, \theta_2 \geq 0 \end{array}\right\} \Longrightarrow \theta_1 x_1 + \theta_2 x_2 \in C$$



convex cone is convex.

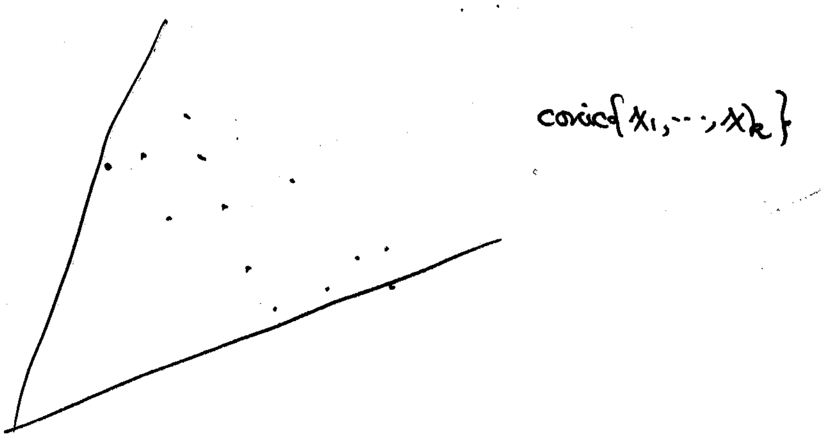
- conic combination of $x_1, \dots, x_k$

$$y = \theta_1 x_1 + \cdots + \theta_k x_k, \quad \theta_i \geq 0 \quad \forall i$$

- conic hull

$$\text{conic } C = \{ \theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C, \ \theta_i \geq 0, \ i = 1, \dots, k \}$$

also the smallest convex cone that contains C.



- examples worth mentioning
  - a subspace is affine, convex, & a convex cone.
  - A ray $\{ x_0 + \theta v \mid \theta \geq 0 \}$, $v \neq 0$, is convex (coz it's a line segment). It's not affine. It's a convex cone if $x_0 = 0$.

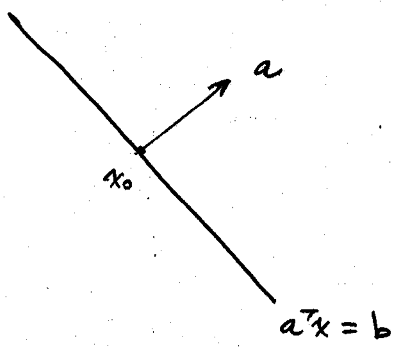
4 Mar. 2008

Hyperplanes and Halfspaces

- A hyperplane is a set of the form

$$\{ x \mid a^T x = b \}$$

where $a \in \mathbb{R}^n$, $a \neq 0$, and $b \in \mathbb{R}$.



- For any $x_0 \in \{ x \mid a^T x = b \}$, the hyperplane can be reexpressed as

$$\{ x \mid a^T (x - x_0) = 0 \}$$

- Recall $\{ x \mid a^T x = 0 \} = \mathcal{N}(a^T) = \mathcal{R}(a)_\perp$. So

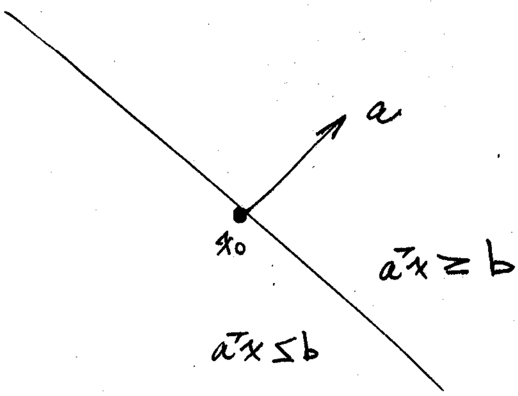
$$\{ x \mid a^T (x - x_0) = 0 \} = x_0 + \mathcal{R}(a)_\perp .$$

- A (closed) halfspace is a set of the form

$$\{x \mid a^T x \leq b\}$$

where $a \neq 0$.

- Q: is halfspace affine?



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Euclidean balls & Ellipsoids

- A Euclidean ball (or just ball) in $\mathbb{R}^n$ has the form

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\}$$

x_c is called the center of the ball

r " " the radius.

- A common, alternate representation of the Euclidean ball is

$$B(x_c, r) = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

- A Euclidean ball is convex: if $\|x_1 - x_c\| \le r$ and $\|x_2 - x_c\| \le r$, $0 \le \theta \le 1$, then

$$\|\theta x_1 + (1-\theta)x_2 - x_c\|_2 = \|\theta(x_1 - x_c) + (1-\theta)(x_2 - x_c)\|_2$$
$$\le \theta\|x_1 - x_c\|_2 + (1-\theta)\|x_2 - x_c\|_2$$
$$\le r.$$

- A related family of convex sets is the ellipsoids

$$\mathcal{E} = \{x \mid (x - x_c)^T P^{-1}(x - x_c) \le 1\}$$

where $P \succ 0$. It can be rewritten as

$$\mathcal{E} = \{x_c + Au \mid \|u\|_2 \le 1\}$$

where A is square.

How?

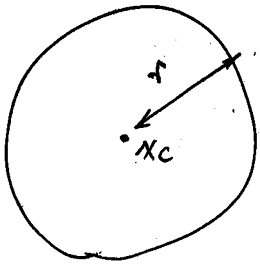
Let $z = x - x_c$.

$$\mathcal{E} = \{x_c + z \mid z^T P^{-1} z \le 1\}.$$

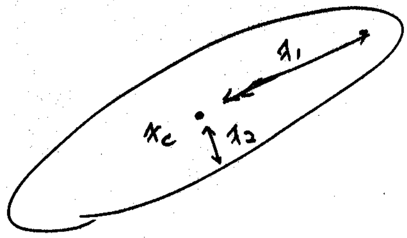
Let $P = A^T A$, so that $P^{-1} = A^{-T}A^{-1}$.

Let $u = A^{-1}z$.

$$\mathcal{E} = \{x_c + Au \mid u^T u \le 1\}.$$



Euclidean ball

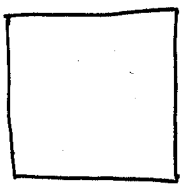


Ellipsoid.

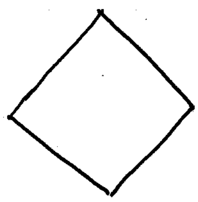
Norm Balls and norm cones

- Suppose $\|.\|$ is any norm on $\mathbb{R}^n$. A norm ball of radius r and center x_c is

$$\{ x \mid \|x - x_c\| \leq r \}.$$



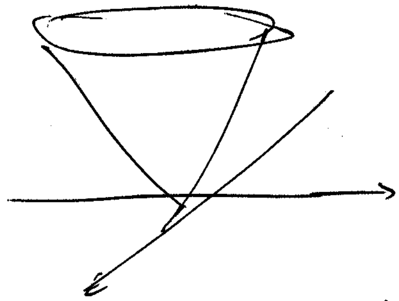
∞-norm ball



1-norm ball

- The norm cone associated with $\|.\|$ is

$$C = \{ (x,t) \mid \|x\| \leq t \} \subseteq \mathbb{R}^{n+1}.$$



2-order cone.

$(x_1, t_1) \in C, \quad (x_2, t_2) \in C, \quad \theta_1, \theta_2 \geq 0$

$$\|\theta_1 x_1 + \theta_2 x_2\| \leq \theta_1 \|x_1\| + \theta_2 \|x_2\|$$
$$\leq \theta_1 t_1 + \theta_2 t_2$$

$$\Rightarrow (\theta_1 x_1 + \theta_2 x_2, \theta_1 t_1 + \theta_2 t_2) \in C.$$

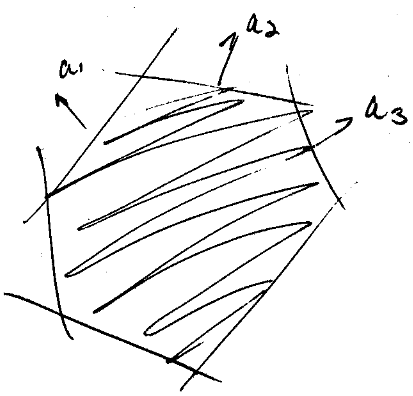
- So, a norm cone is a convex cone.

Polyhedra

- A polyhedron is the solution set of a finite no. of linear equalities & inequalities:

$$\mathcal{P} = \{x \mid a_j^T x \leq b_j,\ j=1,\dots,m,\ c_j^T x = d_j,\ j=1,\dots,p\}$$
$$= \{x \mid Ax \preceq b,\ Cx = d\}, \quad A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}, \ C = \begin{bmatrix} c_1^T \\ \vdots \\ c_p^T \end{bmatrix}$$

where $\preceq$ denotes the vector inequality or componentwise inequality in $\mathbb{R}^n$: $u \preceq v$ means that $u_i \leq v_i$ for all i.



- can be easily shown to be convex. (do it!)

- A bounded polyhedron is called a polytope.

PSD Cone

Mar. 6, 2006

Recall $S^n = \{X \in \mathbb{R}^{n\times n} \mid X = X^T\}$ be the set of sym $n\times n$ matrices.

$S^n_+ = \{X \in S^n \mid X \succeq 0\}$ " " " " PSD " "

$S^n_{++} = \{X \in S^n \mid X \succ 0\}$. " " " " PD " "

- S^n_+ is a convex cone.

$\theta_1, \theta_2 \geq 0, \quad A, B \in S^n_+$

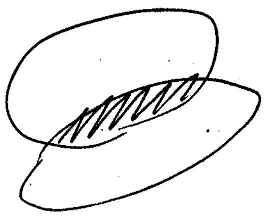
$$\Rightarrow \quad x^T(\theta_1 A + \theta_2 B)x = \theta_1 x^T A x + \theta_2 x^T B x \geq 0$$

, for any $x \in \mathbb{R}^n$.

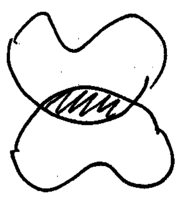
CONVEXITY PRESERVING OPERATIONS

- Intersection

S_1, S_2 convex $\Longrightarrow$ $S_1 \cap S_2$ convex



(But note that $S_1 \cap S_2$ convex $\not\Rightarrow$ S_1, S_2 convex!)



Proof:

Let $x_1, x_2 \in S_1 \cap S_2$. This implies $x_1, x_2 \in S_1$ ~~or~~ and S_2.
For $\theta \in [0, 1]$

$$\theta x_1 + (1-\theta) x_2 \in S_1$$
$$\theta x_1 + (1-\theta) x_2 \in S_2 .$$
$$\Rightarrow \quad \theta x_1 + (1-\theta) x_2 \in S_1 \cap S_2 .$$

- Extension: if S_α is convex for every $\alpha \in \mathcal{A}$, then $\bigcap_{\alpha \in \mathcal{A}} S_\alpha$ is convex.

Example: $S_+^n = \{ X \in S^n \mid X \succeq 0 \}$.

$$\cancel{S_+^n = \bigcap_{z \in \mathbb{R}^n} \{ X \in S^n \mid z^T X z \geq 0 \}}$$

$$S_+^n = \{ X \in S^n \mid z^T X z \geq 0 \text{ for any } z \in \mathbb{R}^n \}$$
$$= \bigcap_{z \in \mathbb{R}^n} \underbrace{\{ X \in S^n \mid z^T X z \geq 0 \}}_{S_z}$$

$$S_z = \{ X \in S^n \mid \langle X, z z^T \rangle \geq 0 \}$$

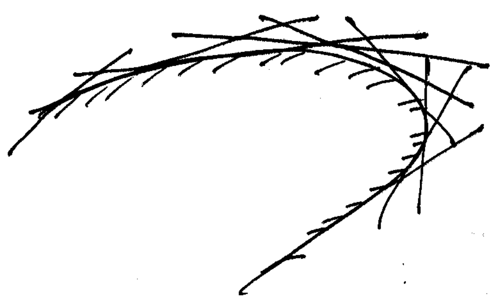
$\Rightarrow$ S_z is a halfspace in S^n!

$\Rightarrow$ S_+^n is convex.

- Interesting: If S is a <u>closed</u> convex set,

$$S = \bigcap \{ \mathcal{H} \mid \mathcal{H} \text{ halfspace}, \ S \subseteq \mathcal{H} \}.$$

S is the intersection of ALL halfspaces that contain it.



Example: — non-ve polynomial

$$C = \{ x \in \mathbb{R}^n \mid x_1 + x_2 t + \cdots x_n t^{n-1} \geq 0, \ t \in [0,1] \}$$

$$= \bigcap_{t \in [0,1]} \{ x \mid \sum_{i=1}^{n} x_i t^{i-1} \geq 0 \}$$

- Affine Functions

- A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is affine if it is a sum of a linear function and a constant

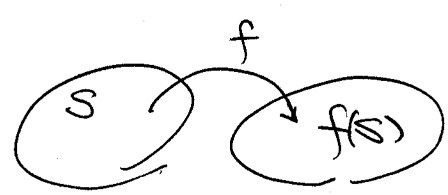
$$f(x) = Ax + b$$

(elaborate linear function)

- If $S \subseteq \mathbb{R}^n$ is convex and $f: \mathbb{R}^n \to \mathbb{R}^m$ is affine, then the image of S under f.

$$f(S) = \{ f(x) \mid x \in S \}$$

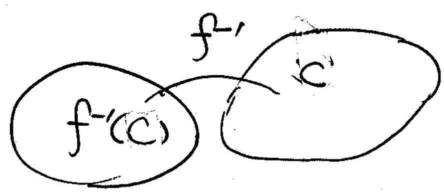
is convex.



- Similarly, the inverse image of C under f

$$f^{-1}(C) = \{ x \mid f(x) \in C \}$$

is convex.



Proof: $y_1, y_2 \in f(S)$. For $\theta \in [0,1]$,

$$\begin{aligned}\theta y_1 + (1-\theta) y_2 &= \theta(Ax_1 + b) + (1-\theta)(Ax_2 + b), \quad \text{for some } x_1, x_2 \in S. \\ &= A\underbrace{(\theta x_1 + (1-\theta) x_2)}_{\in S} + b \\ &\in f(S).\end{aligned}$$

On the other hand, $x_1, x_2 \in f^{-1}(C)$. For $\theta \in [0,1]$,

$$\begin{aligned} f(\theta x_1 + \cancel{\theta}(1-\theta) x_2) &= A(\theta x_1 + (1-\theta)x_2) + b \\ &= \theta(Ax_1 + b) + (1-\theta)(Ax_2 + b) \\ &= \theta f(x_1) + (1-\theta) f(x_2) \in C \end{aligned}$$

$\Rightarrow f^{-1}(C)$ is convex.

- Some important results for affine function property

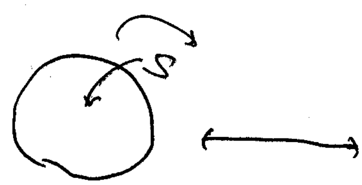
1. $\alpha S = \{\alpha x \mid x \in S\}$ convex (scaling)

2. $S + a = \{x + a \mid x \in S\}$ convex (translation)

3. If $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is convex,

$$\{x_1 \in \mathbb{R}^m \mid (x_1, x_2) \in S \text{ for some } x_2 \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

is convex (projection of S onto some of its coordinates.



4. The sum of convex S_1 & S_2

$$S_1 + S_2 = \{x + y \mid x \in S_1,\ y \in S_2\}$$

To see this, we prove that $S_1 \times S_2$ is convex.
We can find a function that maps $S_1 \times S_2$ to $S_1 + S_2$.

5. Partial sum:

$$S = \{(x, y_1 + y_2) \mid (x, y_1) \in S_1,\ (x, y_2) \in S_2\}$$

<u>Example</u>: Solution set of linear matrix inequality (LMI)

$$A(x) = x_1 A_1 + \cdots + x_n A_n \preceq B$$

(note $A \preceq B$ means that $B - A \succeq 0$ is PSD), where $A_i, B \in S^m$, $x \in \mathbb{R}^n$.

$$\cancel{S}\; C = \{ x \mid A(x) \preceq B \}$$

is convex. Why?

$$z^T B z - \sum_i x_i (z^T A_i z) \geq 0.$$

$$C\; \cancel{S} = \bigcap_{z \in \mathbb{R}^m} \underbrace{\{ x \mid z^T (B - A(x)) z \geq 0 \}}_{\text{halfspace in } \mathbb{R}^n.}$$

Alternative proof:

Let $f : \mathbb{R}^n \to \cancel{\mathbb{R}}\, S^m$ be given by $f(x) = B - A(x)$.

$$\cancel{S} \quad \{ x \mid A(x) \preceq B \} = f^{-1}(S^m_+).$$

$$= \{ x \mid B - A(x) \in S^m_+ \}.$$

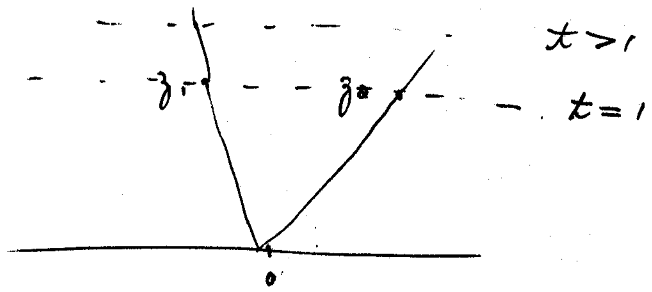
Since S^m_+ is convex, f is affine, $\cancel{S}\; C$ is convex.

LINEAR-FRACTIONAL & PERSPECTIVE FUNCTIONS

- Perspective function: $P: \mathbb{R}^{n+1} \to \mathbb{R}^n$, with domain $\mathbf{dom}\, P = \mathbb{R}^n \times \mathbb{R}_{++}$

$$P(z,t) = z/t.$$

It 'normalizes' z. ~~so that~~



pin-hole camera perspective.

- If $C \subseteq \mathbf{dom}\, P$ is convex, then its image

$$P(C) = \{ P(x) \mid x \in C \}$$

is convex.

- The inverse image of a convex set under the perspective fn. is also convex.

- Linear fractional function (More in convex functions)

$$f(x) = Ax+b/c^Tx+d, \quad \mathrm{dom}\, f = \{x \mid c^Tx+d > 0\}$$

has similar characteristics as affine functions.

More about Convex Sets

Mar. 9
2006

A pt $x \in C$ is an extreme pt. of C if there does not exist $y, z \in C$, $y \neq z$, and $\theta \in (0,1)$ such that

$$x = \theta y + (1-\theta) z .$$

In other words, an extreme pt. cannot be expressed as a convex combination of some pts in C, all of which are different from x.

krein

Krein – Milman Theorem :

Let $C \subseteq \mathbb{R}^n$ be a compact convex set (i.e., closed and bounded). Then C is the closed convex hulls of its extreme points.

Supplement : extreme subset

Suppose K is a non-empty closed convex set.

$A \subseteq K$ is called an extreme subset of K if

A is closed, convex, and

$$\left.\begin{array}{l} x, y \in K \\ \theta x + (1-\theta) y \in A \\ \theta \in (0,1) \end{array}\right. \Rightarrow x, y \in A$$

e.g.