II. Convex Sets — Pt 2.

Proper cones and generalized inequalities

- Recall a set K is a cone if $x \in K \Rightarrow \theta x \in K$, $\theta \geq 0$.
- A cone $K \subseteq \mathbb{R}^n$ is a proper ~~cone~~ if
  - i) K is ~~of~~ convex
  - ii) K is closed
  - iii) K is solid, which means it has nonempty interior.
  - iv) K is pointed; i.e., $x \in K$, $-x \in K \Rightarrow x = 0$.

(The ~~parti~~ partial ordering)

- Generalized inequality associated with a proper cone K

$$x \preceq_K y \iff y - x \in K.$$

$$x \prec_K y \iff y - x \in \operatorname{int} K$$

- Examples

$K = \mathbb{R}^n_+$ (non-ve orthant)

$x \preceq_K y$ means that $x_i \leq y_i$, $i = 1, \cdots, n$.

$K = S^n_+$ (PSD cone)

$X \preceq_K Y$ means that $Y - X$ ~~≥0~~ is PSD.

$$K = \{ c \in \mathbb{R}^n \mid c_1 + c_2 t + \cdots + c_n t^{n-1} \geq 0 \quad \forall\, t \in [0,1] \}$$

(cone of non-ve polynomials on $[0,1]$)

- Careful: $x \not\preceq_K y$ does not imply $x \succeq_K y$!

Sounds true for usual ordering — $x \not\leq y \Rightarrow x > y$.

Consider $K = \mathbb{R}^n_+$. $\begin{pmatrix}1\\3\end{pmatrix} \not\preceq_K \begin{pmatrix}3\\1\end{pmatrix} \not\Rightarrow \begin{pmatrix}1\\3\end{pmatrix} \succeq_K \begin{pmatrix}3\\1\end{pmatrix}$!

- properties of generalized inequalities

i) $x \preceq_K y,\ u \preceq_K v \Rightarrow x+u \preceq_K y+v$ (preserved under addition)

ii) $x \preceq_K y,\ y \preceq_K z \Rightarrow x \preceq_K z$ (transitive)

iii) $x \preceq_K y,\ \alpha \geq 0 \Rightarrow \alpha x \preceq_K \alpha y$ (preserved under non-ve scaling)

iv) $x \preceq_K y,\ y \preceq_K x \Rightarrow x = y$. (antisymmetric)

- properties for strict inequalities

i) $x \prec_K y \Rightarrow x \preceq_K y$

ii) $x \prec_K y$, ~~$y \preceq_K z$~~ $u \preceq_K v \Rightarrow x+u \prec_K y+v$

iii) $x \prec_K y,\ \alpha > 0 \Rightarrow \alpha x \prec_K \alpha y$

iv) $x \not\prec_K x$

v) if $x \prec_K y$, then for small enough u, v

$x+u \prec_K y+v$.

Minimum & Minimal Elements (think about inf., sup. for $C \subseteq \mathbb{R}$)

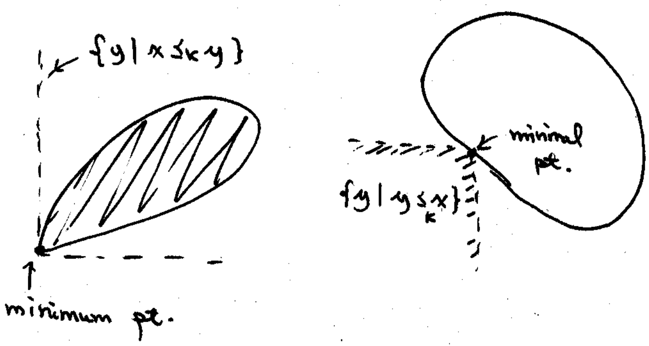
- $x \in S$ is the (unique) minimum element of S if

$$y \in S \Rightarrow x \preceq_K y.$$

The min. element may not exist!

- $x \in S$ is a minimal element of S if

$$\begin{array}{l} y \in S, \\ y \preceq_K x \end{array} \Rightarrow y = x.$$



- $x \in S$ is the minimum element of S iff

$$S \subseteq x + K$$

- $x \in S$ is the minimal element of S iff

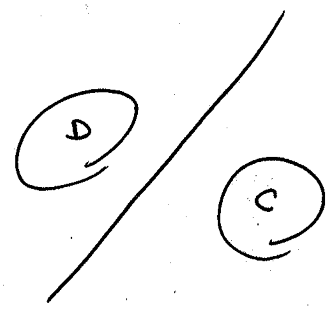
$$(x - K) \cap S = \{x\}$$

Separating Hyperplanes

Suppose that ~~C & D~~ are convex and that $C \cap D = \emptyset$.
$C, D \subseteq \mathbb{R}^n$

Then there exist $a \neq 0$ and b such that

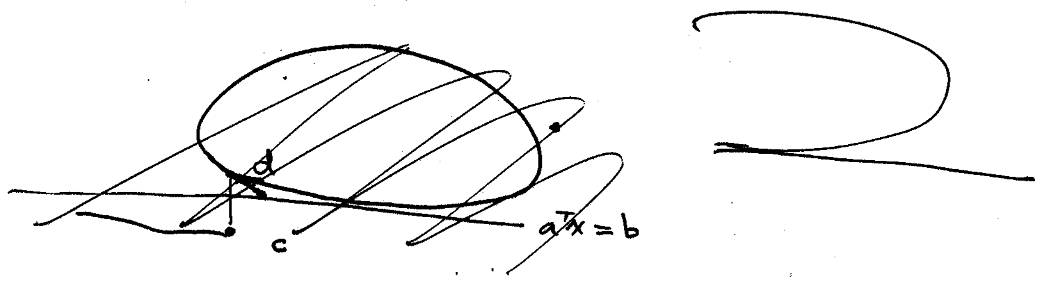
$$a^T x \leq b \Rightarrow x \in C$$
$$a^T x \geq b \Rightarrow x \in D$$



The hyperplane $\{x \mid a^T x = b\}$ is said to be a separating hyperplane for C and D.

- Strict separation: $a^T x < b \ \forall \ x \in C$ and
  $a^T x > b \ \forall \ x \in D$.

Strict separation is not always possible even when C & D are disjoint, and closed. (find an example!



- converse separating hyperplane theorem:

existence of a separating hyperplane $\Rightarrow C \cap D = \emptyset$.

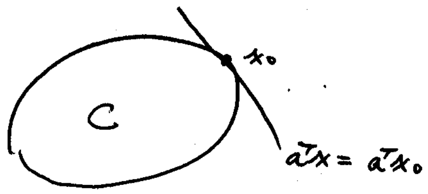
is not true. But $\boxed{\text{Any convex } C, D, \text{ with one of them being open, is disjoint iff there exists a separating hyperplane}}$

Supporting Hyperplanes

Suppose $x_0 \in \text{bd}\, C = \text{cl}\, C \setminus \text{int}\, C$.

If there exist $a \neq 0$ such that $a^T x \leq a^T x_0 \quad \forall\ x \in C$,

then $\{x \mid a^T x = a^T x_0\}$ is called a supporting hyperplane to C at point x_0.



Supporting hyperplane theorem:
For any nonempty convex set C, and for any $x_0 \in \text{bd}\, C$, there exists a supporting hyperplane to C at x_0.

Proof: Suppose $\text{int}\, C \neq \emptyset$. Let $A = \{x_0\}$, where $x_0 \in \text{bd}\, C$, and $B = \text{int}\, C$. By separating hyperplane thm., there exist a hyperplane separating A and B:

$$a^T x \leq b \quad \forall \quad x \in \text{int}\, C$$
$$a^T x_0 \geq b$$

Now, $a^T x \leq a^T x_0$, for $x \in \text{int}\, C$. If C is open ($\text{int}\, C = C$), then the proof is complete. (What to do when C is not open?)