COM521500
Math. Methods for SP I
Lecture 5: Positive Semidefinite Matrices

Quadratic Form

The quantity

$$\mathbf{x}^H \mathbf{A} \mathbf{x}$$

is called the **quadratic form**. It can be expressed as

$$\mathbf{x}^H \mathbf{A} \mathbf{x} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} x_i^* x_k$$

Consider complex-valued, Hermitian $\mathbf{A}$.

The quadratic form $\mathbf{x}^H\mathbf{A}\mathbf{x}$ is real-valued for any $\mathbf{x} \in \mathbb{C}^n$.

Consider real-valued $\mathbf{A}$. Every $\mathbf{A} \in \mathbb{R}^{n\times n}$ can be written as

$$\mathbf{A} = \mathbf{T} + \mathbf{S}$$

where $\mathbf{T} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ is symmetric, and $\mathbf{S} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$ is skew-symmetric; i.e., $\mathbf{S}^T = -\mathbf{S}$.

It can be verified that $\mathbf{x}^T\mathbf{S}\mathbf{x} = 0$ for any $\mathbf{x} \in \mathbb{R}^n$. Hence, $\mathbf{x}^T\mathbf{A}\mathbf{x}$ only depends on the symmetric part of $\mathbf{A}$.

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is symmetric, and

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is skew-symmetric; i.e., $\mathbf{S}^T = -\mathbf{S}$.

It can be verified that $\mathbf{x}^T\mathbf{S}\mathbf{x} = 0$ for any $\mathbf{x}$.

Positive Definite/Semidefinite Matrices

A Hermitian matrix $\mathbf{A} \in \mathbb{C}^n$ is said to be **positive semidefinite (PSD)** if

$$\mathbf{x}^H \mathbf{A} \mathbf{x} \geq 0$$

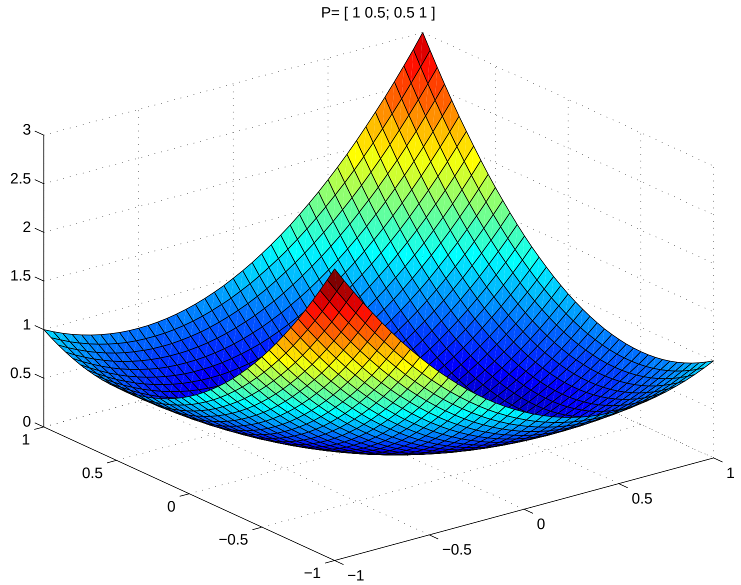
for any $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq \mathbf{0}$.

A Hermitian matrix $\mathbf{A} \in \mathbb{C}^n$ is said to be **positive definite (PD)** if

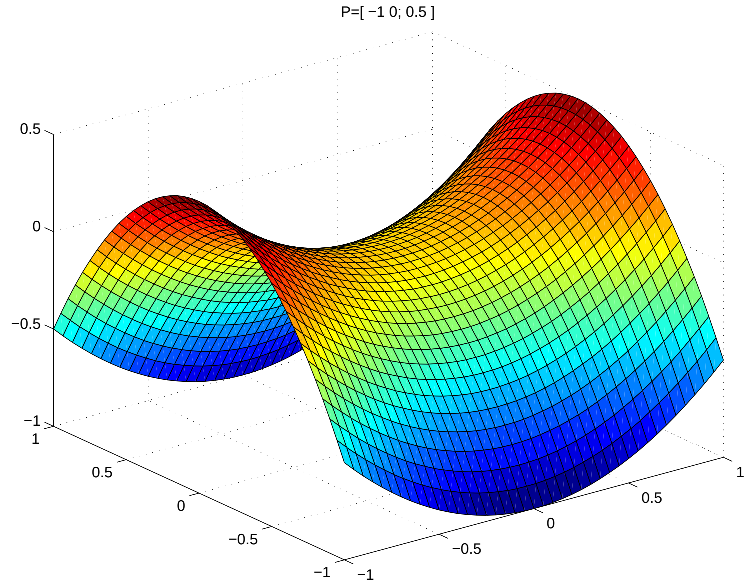
$$\mathbf{x}^H \mathbf{A} \mathbf{x} > 0$$

for any $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq \mathbf{0}$.

A Hermitian matrix that is not PD or PSD is called an **indefinite** matrix.



Quadratic form for a positive definite matrix.



Quadratic form for an indefinite matrix.

PD and PSD matrices are frequently encountered in practice.

For example, the covariance matrix for a random process $\mathbf{x}[n] \in \mathbb{C}^N$

$$\begin{aligned}\mathbf{R}_x &= \mathrm{E}\{\mathbf{x}[n]\mathbf{x}^H[n]\} \\ &= \begin{bmatrix} \mathrm{E}\{|x_1[n]|^2\} & \ldots & \mathrm{E}\{x_1[n]x_N^*[n]\} \\ \vdots & \ddots & \vdots \\ \mathrm{E}\{x_N[n]x_1^*[n]\} & \ldots & \mathrm{E}\{x_N[n]x_1^*[n]\} \end{bmatrix}\end{aligned}$$

is always PSD.

A **principal submatrix** of $\mathbf{A} \in \mathbb{C}^N$, denoted by

$$\mathbf{A}(\{k_1, \ldots, k_r\})$$

where $\{k_1, \ldots, k_r\} \subset \{1, 2, \ldots, n\}$ is an index set, is a matrix obtained by keeping only the k_ith rows and columns of $\mathbf{A}$ for $i = 1, 2, \ldots, r$.

Property 4.1 If $\mathbf{A}$ is PSD, then any principal submatrix is PSD.

Example: Partition

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

If $\mathbf{A}$ is PSD, then $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are PSD.

Property 4.2 Let $\mathbf{A} \in \mathbb{C}^n$ be PD. If $\mathbf{C} \in \mathbb{C}^{n \times m}$ then $\mathbf{C}^H \mathbf{A} \mathbf{C}$ is PSD. Furthermore, $\mathbf{C} \in \mathbb{C}^{n \times m}$ is PD if and only if $\mathrm{rank}(\mathbf{C}) = m$.

Example: Let $\mathbf{x}[n]$ be a WSS process with covariance $\mathbf{R}_x$, and consider another process

$$\mathbf{y}[n] = \mathbf{C}^H \mathbf{x}[n]$$

The covariance of $\mathbf{y}[n]$ is

$$\mathbf{R}_y = \mathbf{C}^H \mathbf{R}_x \mathbf{C}$$

which is PSD.

Theorem 4.1 A Hermitian matrix $\mathbf{A}$ is PSD if and only if all the eigenvalues of $\mathbf{A}$ are non-negative. A Hermitian matrix $\mathbf{A}$ is PD if and only if all the eigenvalues of $\mathbf{A}$ are positive.

It follows that

1. PD matrices are always invertible;
2. $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$ is positive/non-negative for a PD/PSD matrix;
3. $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$ is positive/non-negative for a PD/PSD matrix.

Theorem 4.2 A Hermitian matrix $\mathbf{A}$ can be decomposed into the form

$$\mathbf{A} = \mathbf{B}^H \mathbf{B} \qquad (*)$$

if and only if $\mathbf{A}$ is PD or PSD.

The matrix $\mathbf{B}$ in $(*)$ is called a square root factor. It is not unique.