COM521500
Math. Methods for SP I
Lecture 1: Basic Concepts

Notations

$\mathbb{R}$	real space, or the set of real numbers
$\mathbb{C}$	complex space, or the set of complex numbers
$\mathbb{R}^n, \mathbb{C}^n$	n-dimensional real/complex space
$\mathbf{x}$	column vector
x_i	ith entry of $\mathbf{x}$
$\mathbf{A}$	matrix
a_{ik}	(i,k)th entry of $\mathbf{A}$

$(.)^T$	transpose
$(.)^*$	conjugate
$(.)^H$	Hermitian transpose; i.e., conjugate plus transpose
$\mathrm{tr}(.)$	the trace; i.e.; $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$ where $\mathbf{A} \in \mathbb{C}^{n\times n}$ (or $\mathbf{A} \in \mathbb{R}^{n\times n}$)

Some Concepts about Subspaces

Linear Independence:

A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} \in \mathbb{R}^m$ is linearly independent if

$$\sum_{i=1}^{n} c_i \mathbf{a}_i = \mathbf{0} \iff c_1 = c_2 = \ldots = c_n = 0$$

Subspaces:

A subset $\mathcal{S} \subseteq \mathbb{R}^m$ is called a subspace when the following properties are satisfied:

1. if $\mathbf{x}, \mathbf{y} \in \mathcal{S}$ then $\mathbf{x} + \mathbf{y} \in \mathcal{S}$; and
2. if $\mathbf{x} \in \mathcal{S}$ and $c \in \mathbb{R}$ then $c\mathbf{x} \in \mathcal{S}$.

Span:

The span of a set of vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is the set of all possible linear combinations of $\mathbf{a}_1, \ldots, \mathbf{a}_n$:

$$\mathrm{span}[\mathbf{a}_1, \ldots, \mathbf{a}_n] = \left\{ \mathbf{y} \in \mathbb{R}^m \;\middle|\; \mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{a}_i, \quad c_i \in \mathbb{R} \right\}$$

$\mathrm{span}[\mathbf{a}_1, \ldots, \mathbf{a}_n]$ is a subspace.

A **maximal independent set** is a set of vectors which contains the maximum number of independent vectors spanning the space.

A basis for a subspace is any maximally independent set within the subspace.

Orthogonal complement subspace:

The orthogonal complement subspace of a subspace $\mathcal{S}$ is defined as

$$\mathcal{S}_{\perp} = \left\{ \mathbf{y} \in \mathbb{R}^m \;\middle|\; \mathbf{y}^T \mathbf{x} = 0 \text{ for all } \mathbf{x} \in \mathcal{S} \right\}$$

Range space:

The range space of $\mathbf{A} \in \mathbb{R}^{m \times n}$ is defined as

$$R(\mathbf{A}) = \left\{ \mathbf{y} \in \mathbb{R}^m \;\middle|\; \mathbf{y} = \mathbf{A}\mathbf{x}, \;\text{for } \mathbf{x} \in \mathbb{R}^n \right\}$$

Null space: The null space of $\mathbf{A} \in \mathbb{R}^{m \times n}$ is defined as

$$N(\mathbf{A}) = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \mathbf{A}\mathbf{x} = \mathbf{0} \right\}$$

The **dimension** of a subspace (or the vector space) $\mathcal{S}$, denoted by $\dim \mathcal{S}$, is the maximum number of linear independent vectors that spans the subspace.

Some properties:

Given $\mathbf{A} \in \mathbb{R}^{m \times n}$ where $m \geq n$, $\dim R(\mathbf{A}) \leq n$.

Given $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\dim R(\mathbf{A}) + \dim N(\mathbf{A}) = n$.

Some Matrix Concepts

A matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$ is **nonsingular** if

$$\mathbf{A}\mathbf{x} = \mathbf{0} \Longleftrightarrow \mathbf{x} = \mathbf{0}$$

A square matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ is **invertible** if there exists a matrix $\mathbf{A}^{-1}$, called the inverse of $\mathbf{A}$, such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Determinant:

Consider a square matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$. Define $\mathbf{A}_{ij}$ to be the submatrix obtained from $\mathbf{A}$ by deleting the ith row and jth column of $\mathbf{A}$.

The scalar no. $\det(\mathbf{A}_{ij})$ is called the **minor** associated with a_{ij} of $\mathbf{A}$.

The signed minor $c_{ij} = (-1)^{i+j}\det(\mathbf{A}_{ij})$ is called the **cofactor** of a_{ij}.

Cofactor expansion:

$$\det(\mathbf{A}) = \sum_{j=1}^{m} a_{ij}c_{ij}, \quad \text{for any } i = 1, \ldots, m$$

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Some properties:

$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}), \quad \mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times m}$

$\det(\mathbf{A}) = \det(\mathbf{A}^T)$

$\det(c\mathbf{A}) = c^m \det(\mathbf{A})$

$\det(\mathbf{A}) = 0 \Longleftrightarrow \mathbf{A}$ is singular

For a nonsingular $\mathbf{A}$, $\det(\mathbf{A}^{-1}) = \det(\mathbf{A})^{-1}$

If $\mathbf{B} \in \mathbb{R}^{m \times m}$ is nonsingular then $\det(\mathbf{B}^{-1}\mathbf{AB}) = \det(\mathbf{A})$.

Inverse

Let

$$\tilde{\mathbf{A}} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & & \vdots \\ \vdots & & \ddots & \\ c_{m1} & & & c_{mm} \end{bmatrix}$$

$$\mathbf{A}^{-1} = [\det(\mathbf{A})]^{-1}\tilde{\mathbf{A}}$$

Rank

The rank of a matrix $\mathbf{A}$, denoted by $\mathrm{rank}(\mathbf{A})$, is the maximum no. of linearly independent columns of the matrix. It is also the maximum no. of linearly independent rows of the matrix.

From the definition, we have $\dim R(\mathbf{A}) = \mathrm{rank}(\mathbf{A})$.

A matrix $\mathbf{A}$ is **rank deficient** if $\mathrm{rank}(\mathbf{A}) < \min(m, n)$; otherwise $\mathbf{A}$ is of **full rank**.

Vector Norms, and Inner Product

A **vector norm** is a function $\|.\| : \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfies the following properties:

1. $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$
2. $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$
4. $\|c\mathbf{x}\| = |c|\|\mathbf{x}\|$ for $c \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$

Some examples of vector norms:

The p-norms:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1$$

Special cases of the p-norms:

The 2-norm: $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$

The 1-norm: $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$

The ∞-norm: $\|\mathbf{x}\|_\infty = \max_{i=1,\ldots,n} |x_i|$

The scalar

$$\begin{aligned} < \mathbf{x}, \mathbf{y} > &= \sum_{i=1}^{n} y_i x_i \\ &= \mathbf{y}^T \mathbf{x} \end{aligned}$$

is an **inner product** of $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$.

For the matrix case where $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{m \times n}$,

$$\begin{aligned} < \mathbf{X}, \mathbf{Y} > &= \sum_{i=1}^{m} \sum_{k=1}^{n} y_{ik} x_{ik} \\ &= \mathrm{tr}(\mathbf{X}\mathbf{Y}^T) \end{aligned}$$

Some important inequalities:

Cauchy-Schwartz inequality:

$$|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\|_2\|\mathbf{y}\|_2,$$

and equality holds if and only if $\mathbf{x} = c\mathbf{y}$ for $c \in \mathbb{R}$.

Hölder inequality:

$$|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\|_p\|\mathbf{y}\|_q,$$

where $1/p + 1/q = 1$, $p \geq 1$.

Some final remarks:

We have focused on the case of $\mathbb{R}^n$, for ease of exposition of ideas.

Extensions to the case of $\mathbb{C}^n$ are generally straightforward; i.e., replace '$\mathbb{R}$' by '$\mathbb{C}$'. Sometimes the extensions are subject to minor modifications, though.

For example, an inner product for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ is

$$<\mathbf{x}, \mathbf{y}> = \sum_{i=1}^{n} x_i y_i^* = \mathbf{y}^H\mathbf{x}$$

Likewise, for a complex-valued subspace $\mathcal{S} \subseteq \mathbb{C}^m$,

$$\mathcal{S}_\perp = \left\{ \mathbf{y} \in \mathbb{C}^m \;\middle|\; \mathbf{y}^H\mathbf{x} = 0 \text{ for all } \mathbf{x} \in \mathcal{S} \right\}$$